

Markov Selection and \mathcal{W} -strong Feller for 3D Stochastic Primitive Equations

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Abstract

This paper studies some analytical properties of weak solutions of 3D stochastic primitive equations with periodic boundary conditions. The martingale problem associated to this model is shown to have a family of solutions satisfying the Markov property, which is achieved by means of an abstract selection principle. The Markov property is crucial to extend the regularity of the transition semigroup from small times to arbitrary times. Thus, under a regular additive noise, every Markov solution is shown to have a property of continuous dependence on initial conditions, which follows from employing the weak-strong uniqueness principle and the Bismut-Elworthy-Li formula.

Key words: Markov selection; \mathcal{W} -strong Feller;

1 Introduction

The primitive equations (PEs) derived by Boussinesq approximation are a basic model in the study of large oceanic and atmospheric dynamics. These systems form the analytical core of the most advanced general circulation models. For this reason and due to their challenging nonlinear and anisotropic structure, the PEs have recently received considerable attention from the mathematical community.

The mathematical study of the PEs originated in a series of articles by J.L. Lions, R. Temam, and S. Wang in the early 1990s [17, 18, 19, 20]. They set up the mathematical framework and showed the global existence of weak solutions. For the existence and uniqueness of strong solution, many works are concerned on it. For example, C. Hu, R. Temam and M. Ziane proved the global existence and uniqueness of strong solutions to the viscous primitive equations in thin domains for a large set of initial data whose size depends on the thickness of the domain in [14]. In [12], F. Guillén – González, N. Masmoudi and M.A. Rodríguez-Bellido showed the local existence and uniqueness of strong solutions to the viscous primitive equations for any initial data. C. Cao and E.S. Titi developed a beautiful approach to dealing with the L^6 -norm of the fluctuation \tilde{v} of horizontal velocity and obtained the global well-posedness for the 3D viscous primitive equations in [4]. For the uniqueness of weak solutions, in [16], J. Li and E.S. Titi established some conditional uniqueness of weak solutions to the viscous primitive equations under

periodic boundary conditions, and they proved the global existence and uniqueness of weak solutions with the initial data taken as small L^∞ perturbations of functions in the space $X = \{v \in (L^6(O))^2 | \partial_z v \in (L^2(O))^2\}$.

For the primitive equations in random case, many authors paid attention to it. In [13], B. Guo and D. Huang obtained the existence of universal attractor of strong solution under the assumptions that the momentum equation is driven by an additive stochastic forcing and the thermodynamical equation is driven by a fixed heat source. A. Debussche, N. Glatt-Holtz, R. Temam and M. Ziane established the global well-posedness of strong solution, when the primitive equations are driven by multiplicative random noises in [5]. For the ergodicity, in [6], the authors obtained the existence of global weak solutions, and also obtained the exponential mixing property for the weak solutions which are limits of spectral Galerkin approximations of 3D stochastic primitive equations driven by regular multiplicative noise. For a special case that the stochastic primitive equations are in two space dimensions with small linear multiplicative noise, H. Gao and C. Sun obtained a Wentzell-Freidlin type large deviation principle for by weak convergence method in [10], where they omit the spatial variable y and only take (x, z) into account. Furthermore, they established the Hausdorff dimension of the global attractor is finite in [9]. When the primitive equations are driven by an infinite-dimensional additive fractional noise with Hilbert-space-valued, G. Zhou obtained the existence of random attractor in [26].

As we know, both in deterministic and stochastic case, the uniqueness of weak solutions is an important open problem, which results in many properties of weak solutions disappear. Thus, in order to have a deeper understanding of weak solutions and have some development on their uniqueness, it's natural to explore more properties of them. This article presents a step in this direction. We establish that there exists an almost sure Markov family of the primitive equations forced by multiplicative noise. Furthermore, we obtain that every Markov solution has a property of continuous dependence on the initial conditions (\mathcal{W} -strong Feller) if the primitive equations are driven by a regular additive noise. In comparison with [6], we stress that the main improvement of our paper is that the \mathcal{W} -strong Feller is valid for all Markov solutions and not restricted to solutions which are limits of Galerkin approximations. Moreover, the conditions on the noise here is much weaker than those in [6].

When uniqueness of weak solutions is open, Markov property has no direct meaning but a natural question is the existence of a Markov selection. A sufficient condition for the existence of almost sure Markov selections was provided by B. Goldys, M. Röckner and X. Zhang in [11], where they dealt with an abstract stochastic evolution equations. Here, we apply this sufficient condition to our equations (2.23)-(2.28) and obtain

Theorem 1.1. *Under **Hypothesis H0**, there exists an almost sure Markov family $(P_x)_{x \in H}$ of (2.23)-(2.28).*

The definition of weak solution of (2.23)-(2.28) is in Sect 4.2.

The important part of this paper is to investigate the continuity with respect to the initial conditions (strong Feller property) for the Markov family $(P_x)_{x \in H}$. To achieve this, \mathcal{W} -strong Feller is considered which is weaker than strong Feller in H when \mathcal{W} is a subspace of H . In the past two decades, there

are several works concerned on \mathcal{W} -strong Feller for stochastic evolution equations. In particular, F. Flandoli and M. Romito established an abstract criterion Theorem 5.4 to obtain \mathcal{W} -strong Feller property for Markov selections of 3D Stochastic Navier-Stokes equations in [8]. It says that if a Markov process coincides on a small positive random time with a strong Feller process, then it is strong Feller itself. The idea behind this is to use an approximation by a regularised problem, which has itself strong Feller solutions. For the concrete proof, two key points are needed: weak-strong uniqueness principle and the Bismut-Elworthy-Li formula. It's worth mentioning that this technique is usually applied to handle locally Lipschitz nonlinearities in stochastic equations. To study the strong Feller property of our equations, we will follow the idea of Theorem 5.4 in [8]. Firstly, we introduce an auxiliary Ornstein-Uhlenbeck process Z which is a stationary ergodic solution to a stochastic Stokes equation, then, the approximation process $X_t^{(R)}$ is obtained. To achieve our goal, two steps are needed: the approximation process $X_t^{(R)}$ coincides with the original process on a small positive random time interval and $X_t^{(R)}$ is \mathcal{W} -strong Feller. The second step is more challenging, where we have to control the power of Z to be less than 2 as we want to apply the Fernique's theorem to Z . We overcome this difficulty by making use of $\Lambda \frac{\partial Z}{\partial z}$, which is a key idea in our proof, in that case, $Z \in C([0, T]; D(A))$ is needed, then the noise has to be chosen as $A^{-\frac{5}{4}-\varepsilon_0}$ with $\varepsilon_0 > 0$. Thus, the corresponding \mathcal{W} is equal to $D(A^{\frac{3}{4}+\varepsilon_0})$. In comparison with 3D Navier-Stokes equations, the regularity of the noise here is required to be higher than $A^{-\frac{11}{12}-\varepsilon_0}$ with $\varepsilon_0 > 0$ for 3D Navier-Stokes equations because of strong nonlinear terms $(\int_{-1}^z \nabla_H \cdot v dz') \frac{\partial v}{\partial z}$ and $(\int_{-1}^z \nabla_H \cdot v dz') \frac{\partial T}{\partial z}$ in the primitive equations. Also, the advective structure of the primitive equations leads to a delicate asymmetry in the nonlinear terms, which requires a more refined calculation.

Theorem 1.2. *Assume **Hypothesis H1** holds. Let $(P_x)_{x \in H}$ be the Markov solution of (2.23)-(2.28) and $(P_t)_{t \geq 0}$ be the associated operators on $B_b(H)$ defined as (5.33), then $(P_t)_{t \geq 0}$ is \mathcal{W} -strong Feller.*

The definition of \mathcal{W} -strong Feller is defined in Sect 5.1.

This paper is organized as follows. In Sects. 2 and 3, we introduce the 3D stochastic primitive equations and make formulation of those equations. The abstract Markov selection principle and concrete proof are given in Sect.4. Finally, \mathcal{W} -strong Feller is proved in Sect. 5.

2 Preliminaries

The 3D stochastic primitive equations of the large-scale ocean under a stochastic forcing, in a Cartesian system, are written as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla_H)v + \theta \frac{\partial v}{\partial z} + f k \times v + \nabla_H P - \Delta v = \sigma_1(v, T) \frac{dW_1}{dt}, \quad (2.1)$$

$$\partial_z P + T = 0, \quad (2.2)$$

$$\nabla_H \cdot v + \partial_z \theta = 0, \quad (2.3)$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla_H)T + \theta \frac{\partial T}{\partial z} - \Delta T = \sigma_2(v, T) \frac{dW_2}{dt}, \quad (2.4)$$

where the horizontal velocity field $v = (v^{(1)}, v^{(2)})$, the three-dimensional velocity field $(v^{(1)}, v^{(2)}, \theta)$, the temperature T and the pressure P are unknown functions. f is the Coriolis parameter. k is vertical unit vector. Set $\nabla_H = (\partial_x, \partial_y)$ to be the horizontal gradient operator and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ to be the three dimensional Laplacian. W_1 and W_2 are two independent cylindrical Wiener processes on H_1 and H_2 , respectively. H_1 and H_2 will be defined in Sect. 3.

The spatial variable (x, y, z) belongs to $\mathcal{M} := \mathbb{T}^2 \times (-1, 0)$. For simplicity of the presentation, all the physical parameters (height, viscosity, size of periodic box) are set to 1.

Refer to [2], the boundary value conditions for (2.1)-(2.4) are given by

$$v, \theta \text{ and } T \text{ are periodic in } x \text{ and } y, \quad (2.5)$$

$$(\partial_z v, \theta) |_{z=-1,0} = (0, 0), \quad T |_{z=-1} = 1, \quad T |_{z=0} = 0. \quad (2.6)$$

$$(v, T) |_{t=0} = (v_0, T_0). \quad (2.7)$$

Replacing T and P by $T + z$ and $P - \frac{z^2}{2}$, respectively, then (2.1)-(2.4) with (2.5)-(2.7) is equivalent to the following system

$$\frac{\partial v}{\partial t} + (v \cdot \nabla_H)v + \theta \frac{\partial v}{\partial z} + fk \times v + \nabla_H P - \Delta v = \sigma_1(v, T + z) \frac{dW_1}{dt}, \quad (2.8)$$

$$\partial_z P + T = 0, \quad (2.9)$$

$$\nabla_H \cdot v + \partial_z \theta = 0, \quad (2.10)$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla_H)T + \theta \left(\frac{\partial T}{\partial z} + 1 \right) - \Delta T = \sigma_2(v, T + z) \frac{dW_2}{dt}, \quad (2.11)$$

subject to the boundary and initial conditions

$$v, \theta \text{ and } T \text{ are periodic in } x \text{ and } y, \quad (2.12)$$

$$(\partial_z v, \theta) |_{z=-1,0} = (0, 0), \quad T |_{z=-1, z=0} = 0, \quad (2.13)$$

$$(v, T) |_{t=0} = (v_0, T_0). \quad (2.14)$$

Here, for simplicity, we still denote by T_0 the initial temperature in (2.14), though it is now different from that in (2.7).

Inherent symmetries in the equations show that the solution of the primitive equations on $\mathbb{T}^2 \times (-1, 0)$ with boundaries (2.12)-(2.14) may be recovered by solving the equations with periodic boundary conditions in x, y and z variables on the extended domain $\mathbb{T}^2 \times (-1, 1) := \mathbb{T}^3$, and restricting to $z \in (-1, 0)$.

To see this, consider any solution of (2.8)-(2.11) with boundaries (2.12)-(2.14), we perform that

$$\begin{aligned} v(x, y, z) &= v(x, y, -z), \text{ for } (x, y, z) \in \mathbb{T}^2 \times (0, 1), \\ T(x, y, z) &= -T(x, y, -z), \text{ for } (x, y, z) \in \mathbb{T}^2 \times (0, 1), \\ P(x, y, z) &= P(x, y, -z), \text{ for } (x, y, z) \in \mathbb{T}^2 \times (0, 1), \\ \theta(x, y, z) &= -\theta(x, y, -z), \text{ for } (x, y, z) \in \mathbb{T}^2 \times (0, 1). \end{aligned}$$

We also extend σ_1 in the even fashion and σ_2 in the odd fashion across $\mathbb{T}^2 \times \{0\}$. Hence, we consider the primitive equations on the extended domain $\mathbb{T}^3 = \mathbb{T}^2 \times (-1, 1)$,

$$\frac{\partial v}{\partial t} + (v \cdot \nabla_H)v + \theta \frac{\partial v}{\partial z} + fk \times v + \nabla_H P - \Delta v = \phi(v, T) \frac{dW_1}{dt}, \quad (2.15)$$

$$\partial_z P + T = 0, \quad (2.16)$$

$$\nabla_H \cdot v + \partial_z \theta = 0, \quad (2.17)$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla_H)T + \theta \left(\frac{\partial T}{\partial z} + 1 \right) - \Delta T = \varphi(v, T) \frac{dW_2}{dt}, \quad (2.18)$$

subject to the boundary and initial conditions

$$v, \theta, P \text{ and } T \text{ are periodic in } x, y, z, \quad (2.19)$$

$$v \text{ and } P \text{ are even in } z, \theta \text{ and } T \text{ are odd in } z, \quad (2.20)$$

$$(v, T)|_{t=0} = (v_0, T_0), \quad (2.21)$$

where

$$\phi(v, T) = \sigma_1(v, T + z) \quad \varphi(v, T) = \sigma_2(v, T + z).$$

Because of the equivalent of the above two kinds of boundary and initial conditions, we consider, throughout this paper, the system (2.15)-(2.21) defined on \mathbb{T}^3 . Note that condition (2.20) is a symmetry condition, which is preserved by system (2.15)-(2.18), that is if a smooth solution to system (2.15)-(2.18) exists and is unique, then it must satisfy the symmetry condition (2.20), as long as it is initially satisfied.

Note that the vertical velocity θ can be expressed in terms of the horizontal velocity v , through the incompressibility condition (2.17) and the symmetry condition (2.20), as

$$\theta(t, x, y, z) = \Phi(v)(t, x, y, z) = - \int_{-1}^z \nabla_H \cdot v(t, x, y, z') dz', \quad (2.22)$$

moreover,

$$\int_{-1}^1 \nabla_H \cdot v dz = 0.$$

Supposing that p_b is a certain unknown function at $\Gamma_b := \mathbb{T}^2 \times \{-1\}$, and integrating (2.16) from -1 to z , we have

$$P(x, y, z, t) = p_b(x, y, t) - \int_{-1}^z T(x, y, z', t) dz'.$$

Now, (2.15)-(2.21) can be rewritten as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla_H)v + \Phi(v) \frac{\partial v}{\partial z} + fk \times v + \nabla_H p_b - \int_{-1}^z \nabla_H T dz' - \Delta v = \phi(v, T), \quad (2.23)$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla_H)T + \Phi(v) \frac{\partial T}{\partial z} + \Phi(v) - \Delta T = \varphi(v, T), \quad (2.24)$$

$$\int_{-1}^1 \nabla_H \cdot v dz = 0. \quad (2.25)$$

The boundary and initial conditions for (2.23)-(2.25) are given by

$$v \text{ and } T \text{ are periodic in } x, y \text{ and } z, \quad (2.26)$$

$$v \text{ and } P \text{ are even in } z, \theta \text{ and } T \text{ are odd in } z, \quad (2.27)$$

$$(v, T)|_{t=0} = (v_0, T_0). \quad (2.28)$$

It is easy to know that Markov Selection and \mathcal{W} -strong Feller property for (v, T) of (2.23)-(2.28) implies the same results of the original solution (v, T) of the system (2.15)-(2.21). In the following, we will focus on (2.23)-(2.28).

3 Formulation of (2.23)-(2.28)

3.1 Functional Spaces

Let $\mathcal{L}(K_1; K_2)$ (resp. $\mathcal{L}_2(K_1; K_2)$) be the space of bounded (resp. Hilbert-Schmidt) linear operators from the Hilbert space K_1 to K_2 , the norm is denoted by $\|\cdot\|_{\mathcal{L}(K_1; K_2)}$ ($\|\cdot\|_{\mathcal{L}_2(K_1; K_2)}$). Denote by $|\cdot|_{L^p(\mathbb{T}^2)}$ the norm of $L^p(\mathbb{T}^2)$ and $|\cdot|_p$ the norm of $L^p(\mathbb{T}^3)$ for $p \in \mathbb{N}_+$. In particular, $|\cdot|$ and (\cdot, \cdot) represent the norm and inner product of $L^2(\mathbb{T}^3)$. For the classical Sobolev space $W^{m,2}(\mathbb{T}^3)$, $m \in \mathbb{N}_+$,

$$\begin{cases} W^{m,2}(\mathbb{T}^3) = \{U \in L^2(\mathbb{T}^3) \mid \partial_\alpha U \in L^2(\mathbb{T}^3) \text{ for } |\alpha| \leq m\}, \\ |U|_{W^{m,2}(\mathbb{T}^3)}^2 = \sum_{0 \leq |\alpha| \leq m} |\partial_\alpha U|^2. \end{cases}$$

It's known that $(W^{m,2}(\mathbb{T}^3), |\cdot|_{W^{m,2}(\mathbb{T}^3)})$ is a Hilbert space.

Define working spaces for equations (2.23)-(2.28). Let

$$\begin{aligned} \mathcal{V}_1 &:= \left\{ v \in (C^\infty(\mathbb{T}^3))^2; \int_{-1}^1 \nabla_H \cdot v dz = 0, v \text{ is periodic in } x, y \text{ and even in } z, \int_{\mathbb{T}^3} v dx dy dz = 0 \right\}, \\ \mathcal{V}_2 &:= \left\{ T \in C^\infty(\mathbb{T}^3); T \text{ is periodic in } x, y \text{ and odd in } z, \int_{\mathbb{T}^3} T dx dy dz = 0 \right\}, \end{aligned}$$

V_1 = the closure of \mathcal{V}_1 with respect to the norm $|\cdot|_{W^{1,2}(\mathbb{T}^3)} \times |\cdot|_{W^{1,2}(\mathbb{T}^3)}$,

V_2 = the closure of \mathcal{V}_2 with respect to the norm $|\cdot|_{W^{1,2}(\mathbb{T}^3)}$,

H_1 = the closure of \mathcal{V}_1 with respect to the norm $|\cdot| \times |\cdot|$,

H_2 = the closure of \mathcal{V}_2 with respect to the norm $|\cdot|$,

$$V = V_1 \times V_2, \quad H = H_1 \times H_2.$$

The inner products and norms on V, H are given by

$$\begin{aligned} (U, U_1)_V &= (v, v_1)_{V_1} + (T, T_1)_{V_2}, \\ (U, U_1) &= (v, v_1) + (T, T_1) = (v^{(1)}, v_1^{(1)}) + (v^{(2)}, v_1^{(2)}) + (T, T_1), \\ (U, U)_V^{\frac{1}{2}} &= (v, v)_{V_1}^{\frac{1}{2}} + (T, T)_{V_2}^{\frac{1}{2}}, \quad \|U\|_V = (U, U)_V^{\frac{1}{2}}. \end{aligned}$$

where $U = (v, T)$, $U_1 = (v_1, T_1)$, $v = (v^{(1)}, v^{(2)})$ and $v_1 = (v_1^{(1)}, v_1^{(2)})$.

On the periodic domain \mathbb{T}^3 , it's known that $-\Delta$ is a self-adjoint compact operator, denote by $\{e_n\}_{n=1,2,\dots}$ an eigenbasis and $\{\lambda_n\}_{n=1,2,\dots}$ the corresponding increasing eigenvalue sequence of $-\Delta$. For $s \in \mathbb{R}^+$, define

$$\|f\|_s^2 = \sum_{k=1}^{\infty} |\lambda_k|^s |(f, e_k)|^2$$

and let $H^s(\mathbb{T}^3)$ denote the Sobolev space of all $f \in H$ for which $\|f\|_s$ is finite. It is easy to know that $\|f\|_0 = |f|$ and $\|f\|_1 = |f|_{W^{1,2}(\mathbb{T}^3)}$. For simplicity, denote $\|\cdot\|_1 = \|\cdot\|$. For $s < 0$, define $H^s(\mathbb{T}^3)$ to be the dual of $H^{-s}(\mathbb{T}^3)$. Set $\Lambda = (-\Delta)^{\frac{1}{2}}$, then

$$\|f\|_s^2 = |\Lambda^s f|^2, \quad |\Lambda^s v|^2 = |\Lambda^s v^{(1)}|^2 + |\Lambda^s v^{(2)}|^2, \quad \|U\|_s^2 = |\Lambda^s v|^2 + |\Lambda^s T|^2.$$

3.2 Functionals

Define three bilinear forms $a : V \times V \rightarrow \mathbb{R}$, $a_1 : V_1 \times V_1 \rightarrow \mathbb{R}$, $a_2 : V_2 \times V_2 \rightarrow \mathbb{R}$, and their corresponding linear operators $A : V \rightarrow V'$, $A_1 : V_1 \rightarrow V'_1$, $A_2 : V_2 \rightarrow V'_2$ by setting

$$a(U, U_1) := (AU, U_1) = a_1(v, v_1) + a_2(T, T_1),$$

where

$$a_1(v, v_1) := (A_1 v, v_1) = \int_{\mathbb{T}^3} \left(\nabla_H v \cdot \nabla_H v_1 + \frac{\partial v}{\partial z} \cdot \frac{\partial v_1}{\partial z} \right) dx dy dz,$$

$$a_2(T, T_1) := (A_2 T, T_1) = \int_{\mathbb{T}^3} \left(\nabla_H T \cdot \nabla_H T_1 + \frac{\partial T}{\partial z} \frac{\partial T_1}{\partial z} \right) dx dy dz,$$

for any $U = (v, T)$, $U_1 = (v_1, T_1) \in V$.

Lemma 3.1. (i) *The forms a , a_i ($i = 1, 2$) are coercive, continuous, and therefore, the operators $A : V \rightarrow V'$ and $A_i : V_i \rightarrow V'_i$ ($i = 1, 2$) are isomorphisms. Moreover,*

$$\begin{aligned} a(U, U_1) &\leq C_1 \|U\|_V \|U_1\|_V, \\ a(U, U) &\geq C_2 \|U\|_V^2, \end{aligned}$$

where C_1 and C_2 are two absolute constants (independent of the physically relevant constants Re_i , Rt_i , etc).

(ii) *The isomorphism $A : V \rightarrow V'$ (respectively $A_i : V_i \rightarrow V'_i$ ($i = 1, 2$)) can be extended to a self-adjoint unbounded linear operator on H (respectively on H_i , $i=1,2$), with compact inverse $A^{-1} : H \rightarrow H$ (respectively $A_i^{-1} : H_i \rightarrow H_i$ ($i = 1, 2$)).*

Now, we define three functionals $b : V \times V \times V \rightarrow \mathbb{R}$, $b_i : V_1 \times V_i \times V_i \rightarrow \mathbb{R}$ ($i = 1, 2$) and the associated operators $B : V \times V \rightarrow V'$, $B_i : V_1 \times V_i \rightarrow V'_i$ ($i = 1, 2$) by setting

$$\begin{aligned} b(U, U_1, U_2) &:= (B(U, U_1), U_2) = b_1(v, v_1, v_2) + b_2(v, T_1, T_2), \\ b_1(v, v_1, v_2) &:= (B_1(v, v_1), v_2) = \int_{\mathbb{T}^3} \left[(v \cdot \nabla_H) v_1 + \Phi(v) \frac{\partial v_1}{\partial z} \right] \cdot v_2 dx dy dz, \\ b_2(v, T_1, T_2) &:= (B_2(v, T_1), T_2) = \int_{\mathbb{T}^3} \left[(v \cdot \nabla_H) T_1 + \Phi(v) \frac{\partial T_1}{\partial z} \right] T_2 dx dy dz, \end{aligned}$$

for any $U = (v, T)$, $U_i = (v_i, T_i) \in V$.

Moreover, we define another functional $g : V \times V \rightarrow \mathbb{R}$ and the associated linear operator $G : V \rightarrow V'$ by

$$\begin{aligned} g(U, U_1) &:= (G(U), U_1) \\ &= \int_{\mathbb{T}^3} \left[f(k \times v) \cdot v_1 + (\nabla_H p_b - \int_{-1}^z \nabla_H T dz') \cdot v_1 + \Phi(v) \cdot T_1 \right] dx dy dz. \end{aligned}$$

Finally, using the functionals defined above to obtain the following stochastic evolution equation

$$\begin{cases} dU(t) + AU(t)dt + B(U(t), U(t))dt + G(U(t))dt = \Psi(U(t))dW(t), \\ U(0) = y, \end{cases} \quad (3.29)$$

where

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad \Psi(U) = \begin{pmatrix} \phi(v, T) & 0 \\ 0 & \varphi(v, T) \end{pmatrix}.$$

3.3 Inequalities

Firstly, we recall the integral version of Minkowski inequality for the L^p spaces, $p \geq 1$. Let $O_1 \subset \mathbb{R}^{m_1}$ and $O_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, where m_1 and m_2 are two positive integers. Suppose that $f(\xi, \eta)$ is measurable over $O_1 \times O_2$. Then

$$\left[\int_{O_1} \left(\int_{O_2} |f(\xi, \eta)| d\eta \right)^p d\xi \right]^{1/p} \leq \int_{O_2} \left(\int_{O_1} |f(\xi, \eta)|^p d\xi \right)^{1/p} d\eta.$$

Lemma 3.2. ([4]) If $v_1 \in H^1(\mathbb{T}^3)$, $v_2 \in H^2(\mathbb{T}^3)$, $v_3 \in H^1(\mathbb{T}^3)$, then

- (i) $|\int_{\mathbb{T}^3} (v_1 \cdot \nabla_H) v_2 \cdot v_3 dx dy dz| \leq c |\nabla_H v_2| |v_3| |v_1|_6 \leq c |\nabla_H v_2| |v_3|^{\frac{1}{2}} |\nabla_H v_3|^{\frac{1}{2}} |\nabla_H v_1|,$
- (ii) $|\int_{\mathbb{T}^3} \Phi(v_1) v_{2z} \cdot v_3 dx dy dz| \leq c |\nabla_H v_1| |v_3|^{\frac{1}{2}} |\nabla_H v_3|^{\frac{1}{2}} |\partial_z v_2|^{\frac{1}{2}} |\nabla_H \partial_z v_2|^{\frac{1}{2}}.$

4 Markov Selection

In the following, we will introduce Markov selection for stochastic evolution equations using the same notations as [11].

4.1 Preliminaries

Let $(\mathbb{X}, \rho_{\mathbb{X}})$ be a polish space and set $\Omega := C([0, \infty); \mathbb{X})$. Denote by \mathcal{B} the Borel σ -field of Ω and by $Pr(\Omega)$ the set of all probability measures on (Ω, \mathcal{B}) . Define the canonical process $\xi : \Omega \rightarrow \mathbb{X}$ as

$$\xi_t(\omega) = \omega(t).$$

For fixed $t \geq 0$, let $\Omega^t := C([t, \infty); \mathbb{X})$ be the space of all continuous functions from $[t, \infty)$ to \mathbb{X} with the metric

$$\rho^t(x, y) := \sum_{m=\lfloor t \rfloor + 1}^{\infty} \frac{1}{2^m} \left(\sup_{s \in [t, m]} \rho_{\mathbb{X}}(x(s), y(s)) \wedge 1 \right)$$

where $\lfloor t \rfloor$ denotes the integer part of t . Then (Ω^t, ρ^t) is a Polish space. For $s \geq t$, define the σ -algebra \mathcal{B}_s^t on Ω^t by $\mathcal{B}_s^t := \sigma[\xi_r : t \leq r \leq s]$, and write $\mathcal{B}^t := \bigcup_{s \geq t} \mathcal{B}_s^t$. Thus, we have a measurable space with filtration $(\Omega^t, \mathcal{B}^t, (\mathcal{B}_s^t)_{s \geq t})$. If $t = 0$, we simply write $(\Omega, \mathcal{B}, (\mathcal{B}_s)_{s \geq 0})$. Finally, define the map $\Phi_t : \Omega \rightarrow \Omega^t$ defined by

$$\Phi_t(\omega)(s) := \omega(s - t), \quad s \geq t,$$

which establishes a measurable isomorphism between $(\Omega, \mathcal{B}, (\mathcal{B}_s)_{s \geq 0})$ and $(\Omega^t, \mathcal{B}^t, (\mathcal{B}_s^t)_{s \geq t})$.

Given $P \in Pr(\Omega)$ and $t > 0$, denote $\omega \mapsto P|_{\mathcal{B}_t}^{\omega} : \Omega \rightarrow Pr(\Omega^t)$ a regular conditional probability distribution of P on \mathcal{B}_t . Since Ω is a Polish space and every σ -field \mathcal{B}_t is finitely generated, such a function exists and is unique, up to P -null sets. In particular,

$$P|_{\mathcal{B}_t}^{\omega}[\xi_t = \omega(t)] = 1$$

for all $\omega \in \Omega$, and if $A \in \mathcal{B}_t$ and $B \in \mathcal{B}^t$,

$$P(A \cap B) = \int_A P|_{\mathcal{B}_t}^{\omega}(B) P(d\omega).$$

Refer to [8], we introduce the following definitions.

Definition 4.1. Given a family $(P_x)_{x \in H}$ of probability measures in $Pr(\Omega)$, the Markov property can be stated as

$$P_x|_{\mathcal{B}_t}^{\omega} = \Phi(t)P_{\omega(t)}, \quad \text{for } P_x - a.e. \omega \in \Omega,$$

for each $x \in H$ and for all $t \geq 0$.

Definition 4.2. The family $(P_x)_{x \in H}$ has the almost sure Markov property if for each $x \in H$, there is a set $\Gamma \subset (0, \infty)$ with null Lebesgue measure, such that

$$P_x|_{\mathcal{B}_t}^{\omega} = \Phi(t)P_{\omega(t)}, \quad \text{for } P_x - a.e. \omega \in \Omega,$$

for all $t \notin \Gamma$.

4.2 A General Criterion

Let \mathbb{H} be a separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\| \cdot \|_{\mathbb{H}}$. Let \mathbb{X}, \mathbb{U} be two separable and reflexive Banach spaces with norms $\| \cdot \|_{\mathbb{X}}$ and $\| \cdot \|_{\mathbb{U}}$, such that

$$\mathbb{U} \subset \mathbb{H} \subset \mathbb{X}$$

continuously and densely. If we identify the dual of \mathbb{H} with itself, then we get

$$\mathbb{X}^* \subset \mathbb{H}^* \simeq \mathbb{H} \subset \mathbb{X}.$$

The dual pair between \mathbb{X} and \mathbb{X}^* is denoted by

$${}_{\mathbb{X}}\langle x, y \rangle_{\mathbb{X}^*}, \quad x \in \mathbb{X}, \quad y \in \mathbb{X}^*.$$

We remark that if $x \in \mathbb{H}$, then

$${}_{\mathbb{X}}\langle x, y \rangle_{\mathbb{X}^*} = \langle x, y \rangle_{\mathbb{H}}.$$

Let \mathcal{E} be a fixed countable dense subset of \mathbb{X}^* which will be chosen in each case and $(W(t))_{t \geq 0}$ be a cylindrical Brownian motion in another separable Hilbert space $(\mathbb{Y}, \| \cdot \|_{\mathbb{Y}})$ with identity covariance. Consider the following evolution equation:

$$dX(t) = \mathcal{A}(X(t))dt + \mathcal{R}(X(t))dW(t), \quad t \geq 0, \quad X(0) = x_0 \in \mathbb{H}, \quad (4.30)$$

where $\mathcal{A} : \mathbb{U} \rightarrow \mathbb{X}$ is $\mathcal{B}(\mathbb{U})/\mathcal{B}(\mathbb{X})$ -measurable and $\mathcal{R} : \mathbb{U} \rightarrow \mathcal{L}_2(\mathbb{Y}; \mathbb{H})$ is $\mathcal{B}(\mathbb{U})/\mathcal{B}(\mathcal{L}_2(\mathbb{Y}; \mathbb{H}))$ -measurable.

Definition 4.3. [11] Let $x_0 \in \mathbb{H}$. A probability measure $P \in \text{Pr}(\Omega)$ is called a martingale solution of (4.30) with initial value x_0 , if it satisfies

(M1) $P(X(0) = x_0) = 1$ and for any $n \in \mathbb{N}_+$

$$P\left\{X \in \Omega : \int_0^n \|\mathcal{A}(X(s))\|_{\mathbb{X}} ds + \int_0^n \|\mathcal{R}(X(s))\|_{\mathcal{L}_2(\mathbb{Y}; \mathbb{H})}^2 ds < +\infty\right\} = 1;$$

(M2) for every $l \in \mathcal{E}$, the process

$$M_l(t, X) := \langle X(t), l \rangle_{\mathbb{X}^*} - \int_0^t \langle \mathcal{A}(X(s)), l \rangle_{\mathbb{X}^*} ds$$

is a continuous square integrable \mathcal{B}_t -martingale with respect to P , whose quadratic variation process is given by

$$\langle M_l \rangle(t, X) := \int_0^t \|\mathcal{R}^*(X(s))(l)\|_{\mathbb{Y}}^2 ds,$$

where the asterisk denote the adjoint operator of $\mathcal{R}(X(s))$;

(M3) for any $p \in \mathbb{N}$, there exists a continuous positive real function $t \mapsto C_{t,p}$ (only depending on p and \mathcal{A}, \mathcal{R}), a lower semi-continuous functional $\mathcal{N}_p : \mathbb{U} \rightarrow [0, \infty]$, and a Lebesgue null set $\mathbb{T}_p \subset (0, \infty)$ such that for all $0 \leq s \notin \mathbb{T}_p$ and all $t \geq s$

$$\mathbb{E}^P \left(\sup_{r \in [s, t]} \|X(r)\|_{\mathbb{H}}^{2p} + \int_s^t \mathcal{N}_p(X(r)) dr \middle| \mathcal{B}_s \right) \leq C_{t-s, p} \cdot (\|X(s)\|_{\mathbb{H}}^{2p} + 1). \quad (4.31)$$

Remark 1. The above definition of martingale solution is in the sense of Stroock and Varadhan's martingale problem in [24], which is weaker than that in [6].

In [11], the authors give a sufficient conditions on \mathcal{A} and \mathcal{R} to obtain Markov family $\{P_{x_0}\}_{x_0 \in \mathbb{H}}$ for (4.30). For this purpose, they firstly introduced the following function class \mathcal{U}^q , $q \geq 1$: A lower semi-continuous function $\mathcal{N} : \mathbb{U} \rightarrow [0, \infty]$ belong to \mathcal{U}^q if $\mathcal{N}(y) = 0$ implies $y = 0$, and

$$\mathcal{N}(cy) \leq c^q \mathcal{N}(y), \quad \forall c \geq 0, y \in \mathbb{U}.$$

and

$$\{y \in \mathbb{U} : \mathcal{N}(y) \leq 1\} \text{ is relatively compact in } \mathbb{U}.$$

The assumptions on \mathcal{A} and \mathcal{R} are given as follows:

(C1) (Demi-Continuity) For any $x \in \mathbb{X}^*$, if y_n strong converges to y in \mathbb{U} , then

$$\lim_{n \rightarrow \infty} \mathbb{X} \langle \mathcal{A}(y_n), x \rangle_{\mathbb{X}^*} = \mathbb{X} \langle \mathcal{A}(y), x \rangle_{\mathbb{X}^*},$$

and

$$\lim_{n \rightarrow \infty} \|\mathcal{R}^*(y_n)(x) - \mathcal{R}^*(y)(x)\|_{\mathbb{Y}} = 0.$$

(C2) (Coercivity Condition) There exist $\lambda_1 \geq 0$ and $\mathcal{N}_1 \in \mathcal{U}^q$ for some $q \geq 2$ such that for all $x \in \mathbb{X}^*$

$$\mathbb{X} \langle \mathcal{A}(x), x \rangle_{\mathbb{X}^*} \leq -\mathcal{N}_1(x) + \lambda_1(1 + \|x\|_{\mathbb{H}}^2).$$

(C3) (Growth Condition) There exist $\lambda_2, \lambda_3, \lambda_4 > 0$ and $\gamma' \geq \gamma > 1$ such that for all $x \in \mathbb{U}$

$$\begin{aligned} \|\mathcal{A}(x)\|_{\mathbb{X}}^{\gamma} &\leq \lambda_2 \mathcal{N}_1(x) + \lambda_3(1 + \|x\|_{\mathbb{H}}^{\gamma'}), \\ \|\mathcal{R}(x)\|_{\mathcal{L}_2(\mathbb{Y}; \mathbb{H})}^2 &\leq \lambda_4(1 + \|x\|_{\mathbb{H}}^2), \end{aligned}$$

where \mathcal{N}_1 is as in (C2).

Theorem 4.1. ([11]) Assume (C1) – (C3) hold, for each $x_0 \in \mathbb{H}$, there exists a martingale solution $P \in \text{Pr}(\Omega)$ starting from x_0 to (4.30) in the sense of Definition 4.3.

Then, the main result is

Theorem 4.2. ([11]) Under (C1) – (C3), there exists an almost sure Markov family $(P_x)_{x \in \mathbb{H}}$ to (4.30).

4.3 Proof of Theorem 1.1

In this part, we will use Theorem 4.2 to get an almost surely Markov family $\{P_x\}_{x \in H}$ for (3.29). Firstly, define the operator \mathcal{A} and \mathcal{R} as follows:

$$\begin{aligned}\mathcal{A}(y) &:= Ay + B(y, y) + G(y), \\ \mathcal{R}(y) &:= \Psi(y) \text{ for } y \in C^\infty(\mathbb{T}^3).\end{aligned}$$

Here, for our equation (3.29), we choose

$$\mathbb{U} = H^1(\mathbb{T}^3), \quad \mathbb{Y} = \mathbb{H} = H, \quad \mathbb{X} = (H^3(\mathbb{T}^3))^*, \quad \mathbb{X}^* = H^3(\mathbb{T}^3),$$

then \mathbb{X} is a Hilbert space and $\mathbb{X}^* \subset \mathbb{U}$ compactly. Moreover, the covariance operator Ψ is assumed to satisfy

Hypothesis H0 (i) $\Psi : H^1(\mathbb{T}^3) \rightarrow \mathcal{L}_2(H; H)$ is a continuous and bounded Lipschitz mapping, i.e.

$$\|\Psi(y)\|_{\mathcal{L}_2(H; H)}^2 \leq \lambda_0 |y|^2 + \rho \quad y \in H^1(\mathbb{T}^3),$$

for some constants $\lambda_0 \geq 0, \rho \geq 0$.

(ii) If $y, y_n \in H^1(\mathbb{T}^3)$, such that y_n strongly converges to y in $H^1(\mathbb{T}^3)$, then for any $x \in C^\infty(\mathbb{T}^3)$,

$$|\Psi(y_n)^*(x) - \Psi(y)^*(x)|_H \rightarrow 0 \quad n \rightarrow \infty.$$

By Lemma 4.1 below, \mathcal{A} can be extended to an operator $\mathcal{A} : H^1(\mathbb{T}^3) \rightarrow \mathbb{X}$. For $y \notin H^1(\mathbb{T}^3)$, $\mathcal{A}(y) := \infty$.

Lemma 4.1. *For any $y_1, y_2 \in C^\infty(\mathbb{T}^3)$,*

$$\begin{aligned}\|Ay_1 - Ay_2\|_{\mathbb{X}} &\leq C_1 |y_1 - y_2|, \\ \|B(y_1, y_1) - B(y_2, y_2)\|_{\mathbb{X}} &\leq C_2 (\|y_1\| + \|y_2\|) \|y_1 - y_2\|, \\ \|G(y_1) - G(y_2)\|_{\mathbb{X}} &\leq C_3 |y_1 - y_2|.\end{aligned}$$

for constants C_1, C_2, C_3 . In particular, the operator $\mathcal{A} : C^\infty(\mathbb{T}^3) \rightarrow \mathbb{X}$ extends to an operator $\mathcal{A} : H^1(\mathbb{T}^3) \rightarrow \mathbb{X}$ by continuity.

Proof of Lemma 4.1 We only prove the second assertion, the first and third estimates can be obtained by Hölder inequality. Refer to [18],

$$\|B(y, y_1)\|_{-3} \leq C |y| \|y_1\|.$$

Then,

$$\begin{aligned}\|B(y_1, y_1) - B(y_2, y_2)\|_{\mathbb{X}} &= \sup_{x \in C^\infty(\mathbb{T}^3): \|x\|_3 \leq 1} |\langle B(y_1, y_1) - B(y_2, y_2), x \rangle| \\ &= \sup_{x \in C^\infty(\mathbb{T}^3): \|x\|_3 \leq 1} |\langle B(y_1, y_1 - y_2) - B(y_1 - y_2, y_2), x \rangle| \\ &\leq C_2 (\|y_1\| + \|y_2\|) \|y_1 - y_2\|.\end{aligned}$$

■

In order to use Theorem 4.2, define the functional \mathcal{N}_1 on \mathbb{U} as follows:

$$\mathcal{N}_1(y) := \begin{cases} \|y\|^2, & \text{if } y \in H^1(\mathbb{T}^3), \\ +\infty, & \text{otherwise.} \end{cases}$$

It is obvious that $\mathcal{N}_1 \in \mathcal{U}^2$.

Proof of Theorem 1.1 By Theorem 4.2, we only need to check (C1)-(C3) for \mathcal{A} and \mathcal{R} .

(I) The demi-continuity condition (C1) holds by Lemma 4.1 and **Hypothesis H0**.

(II) The coercivity condition (C2) follows since

$$\langle B(y, y), y \rangle = 0,$$

then, by Young inequality, we have

$$\begin{aligned} {}_{H^{-3}(\mathbb{T}^3)}\langle \mathcal{A}(y), y \rangle_{H^3(\mathbb{T}^3)} &= {}_{H^{-3}(\mathbb{T}^3)}\langle Ay + B(y, y) + G(y), y \rangle_{H^3(\mathbb{T}^3)} \\ &= -\|y\|^2 + C|y|\|y\| \\ &\leq -\|y\|^2 + \frac{1}{2}\|y\|^2 + C|y|^2 \\ &\leq -\frac{1}{2}\|y\|^2 + \lambda_1(1 + |y|^2). \end{aligned}$$

(III) The growth condition (C3) is clear since by Lemma 4.1, it gives

$$\|\mathcal{A}(y)\|_{-3}^2 \leq \lambda_1\|y\|^2 + \lambda_2(1 + |y|^2),$$

and by **Hypothesis H0**, we have

$$\|\mathcal{R}(y)\|_{\mathcal{L}_2(H;H)}^2 \leq \lambda_3(1 + |y|^2).$$

■

Remark 2. By Theorem 4.1, for any $x_0 \in H$, there exists a martingale solution $P_{x_0} \in \text{Pr}(\Omega)$ to (3.29) in the sense of Definition 4.3. Refer to [11] and [22], we know that P_{x_0} is obtained by means of maximisation.

5 \mathcal{W} -strong Feller

In this section, we apply the abstract result (Theorem 5.4 in [8]) to obtain that every Markov selection in Sect. 4 has \mathcal{W} -strong Feller property.

5.1 Preliminaries

Firstly, we recall the following important lemma ([21], Lemma A.4):

Lemma 5.1. *Suppose that $s > 0$ and $p \in (1, \infty)$. If $f, g \in C^\infty(\mathbb{T}^3)$, then*

$$|\Lambda^s(fg)|_p \leq C(|f|_{p_1} |\Lambda^s g|_{p_2} + |g|_{p_3} |\Lambda^s f|_{p_4}),$$

with $p_i \in (1, \infty]$, $i = 1, \dots, 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Remark 3. $|g|_{p_3} |\Lambda^s f|_{p_4}$ can be equal to $|f|_{p_1} |\Lambda^s g|_{p_2}$ by choosing suitable parameters p_3 and p_4 , in that case, we only write one of them.

We will also use the following Sobolev inequality ([23], Chapter V):

Lemma 5.2. *Suppose that $q > 1, p \in [q, \infty)$ and*

$$\frac{1}{p} + \frac{\sigma}{3} = \frac{1}{q}.$$

If $\Lambda^\sigma f \in L^q(\mathbb{T}^3)$, then $f \in L^p(\mathbb{T}^3)$ and there is a constant $C \geq 0$ independent of f such that

$$|f|_p \leq C |\Lambda^\sigma f|_q.$$

We shall use as well the following interpolation inequality ([15], (5.5)).

Lemma 5.3. *For $f \in C^\infty(\mathbb{T}^3)$, we have*

$$\|f\|_s \leq C \|f\|_{s_1}^{\frac{s_2-s}{s_2-s_1}} \|f\|_{s_2}^{\frac{s-s_1}{s_2-s_1}}, \quad s_1 < s < s_2.$$

Refer to the appendix of [3], we have the following lemma.

Lemma 5.4. *For any v, T and $\omega \in C^\infty(\mathbb{T}^3)$,*

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} \Phi(v) \frac{\partial T}{\partial z} \omega \right| \\ & \leq \int_{\mathbb{T}^2} \left(\int_{-1}^1 |\nabla_H v| dz' \right) \left(\int_{-1}^1 \left| \frac{\partial T}{\partial z} \omega \right| dz \right) dx dy \\ & \leq C \int_{\mathbb{T}^2} \left(\int_{-1}^1 |\nabla_H v| dz' \right) \left(\int_{-1}^1 \left| \frac{\partial T}{\partial z} \right|^2 dz \right)^{\frac{1}{2}} \left(\int_{-1}^1 |\omega|^2 dz \right)^{\frac{1}{2}} dx dy \\ & \leq C \|\omega\| \|\nabla_H v\|_{s_1} \|T\|_{s_2}, \end{aligned}$$

where $s_1 + s_2 = 1$.

At last, we introduce the definition of \mathcal{W} -strong Feller.

Definition 5.1. (\mathcal{W} -strong Feller) *A given semigroup $(P_t)_{t \geq 0}$ on $B_b(H)$ is \mathcal{W} -strong Feller, if for any $t > 0$ and $\psi \in B_b(H)$, $P_t \psi \in C_b(\mathcal{W})$.*

5.2 \mathcal{W} Space and Hypothesis

For any $\varepsilon_0 > 0$ and set

$$\mathcal{W} = D(A^{\frac{3}{4}+\varepsilon_0}), \quad |U|_{\mathcal{W}} = |v|_{\mathcal{W}} + |T|_{\mathcal{W}} = |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| + |\Lambda^{\frac{3}{2}+2\varepsilon_0} T|.$$

In this section, we choose

$$\Omega := C([0, \infty); H^{-\beta})$$

for some $\beta > 3$ and \mathcal{B} denote the Borel σ -algebra on Ω . We assume the noise is additive, nondegenerate and regular. Concretely,

Hypothesis H1 There are an isomorphism Q_0 of H and a number $\alpha_0 = \frac{1}{2} + \varepsilon_0$ such that

$$\Psi = Q^{\frac{1}{2}} = A^{-\frac{3}{4}-\alpha_0} Q_0^{\frac{1}{2}} = A^{-\frac{5}{4}-\varepsilon_0} Q_0^{\frac{1}{2}},$$

where the covariant $Q: H \rightarrow H$ is a symmetric non-negative trace-class operator on H .

Remark 4. Firstly, we notice that **Hypothesis H1** implies **Hypothesis H0**. Indeed, the operator $A^{-\frac{3}{4}-\varepsilon}$ is Hilbert-Schmidt in H , for every $\varepsilon > 0$. Moreover, $A^{-\frac{3}{4}-\varepsilon} Q_0^{\frac{1}{2}} W(t)$ is a Brownian motion in H , for every $\varepsilon > 0$ and every isomorphism Q_0 of H , where $W(t)$ is a cylindrical Wiener process on H . In conclusion, $A^{-\frac{3}{4}-\alpha_0} Q_0^{\frac{1}{2}} W(t)$ is a Brownian motion in $D(A^\alpha)$ for every $\alpha_0 > \alpha > 0$.

In the following, we will consider equations (3.29) in the following abstract form:

$$\begin{cases} dU(t) + AU(t)dt + B(U(t), U(t))dt + G(U(t))dt = Q^{\frac{1}{2}}dW(t), \\ U(0) = y. \end{cases} \quad (5.32)$$

Remark 5. Under **Hypothesis H1**, in [13], the authors have proved that for $y \in V$, there exists a unique strong solution $U = (v, T)$. However, for $y \in H$, the uniqueness of the weak solution is still open, hence, we have to deal with the selected Markov process.

For $y \in H$, let P_y denote the law of the corresponding solution $U(\cdot, y)$ to (5.32). Since **Hypothesis H1** implies **Hypothesis H0**, by Theorem 1.1, the measures $P_y, y \in H$ form a Markov process. Let $(P_t)_{t \geq 0}$ be the associated transition semigroup on $B_b(H)$, defined as

$$P_t(\varphi)(y) := \mathbb{E}[\varphi(U(t, y))] \quad \forall y \in H, \quad \forall \varphi \in B_b(H). \quad (5.33)$$

5.3 Proof of Theorem 1.2

For the proof, we shall use Theorem 5.4 in [8], which is an abstract result to prove the strong Feller property of Markov selection. In order to achieve this, we follow the idea of Theorem 5.11 in [8] to construct $P_y^{(R)}$. We introduce an equation which differs from the original one by a cut-off only, so that with large probability they have the same trajectories on a small random time interval. Consider

$$dU(t) + AU(t)dt + \chi_R(|U|_{\mathcal{W}}^2) [B(U(t), U(t)) + G(U(t))]dt = Q^{\frac{1}{2}}dW(t), \quad (5.34)$$

where $\chi_R: \mathbb{R} \rightarrow [0, 1]$ is of class C^∞ such that $\chi_R(|U|) = 1$ if $|U| \leq R$, and $\chi_R(|U|) = 0$ if $|U| \geq R + 1$ and its first derivative bounded by 1.

Theorem 5.1. (Weak-strong uniqueness) Suppose **Hypothesis H1** holds. Then for every $y \in \mathcal{W}$, equation (5.34) has a unique martingale solution $P_y^{(R)}$, with

$$P_y^{(R)}[C([0, \infty); \mathcal{W})] = 1.$$

Let $\tau_R : \Omega \rightarrow [0, \infty]$ be defined by

$$\tau_R(\omega) := \inf\{t \geq 0 : |\omega(t)|_{\mathcal{W}}^2 \geq R\}$$

and $\tau_R(\omega) := \infty$ if this set is empty. If $y \in \mathcal{W}$ and $|y|_{\mathcal{W}}^2 < R$, then

$$\lim_{\varepsilon \rightarrow 0} P_{y+h}^{(R)}[\tau_R \geq \varepsilon] = 1, \text{ uniformly in } h \in \mathcal{W}, |h|_{\mathcal{W}} < 1. \quad (5.35)$$

Moreover,

$$\mathbb{E}^{P_y^{(R)}}[\varphi(\omega_t)I_{[\tau_R \geq t]}] = \mathbb{E}^{P_y}[\varphi(\omega_t)I_{[\tau_R \geq t]}] \quad (5.36)$$

for every $t \geq 0$ and $\varphi \in B_b(H)$.

Proof of Theorem 5.1 Let Z be the solution to

$$dZ(t) + AZ(t)dt = Q^{\frac{1}{2}}dW(t),$$

with the initial data $Z(0) = 0$ and let $X_y^{(R)}$ be the solution to the auxiliary problem

$$\frac{dX^{(R)}(t)}{dt} + AX^{(R)}(t) + \chi_R(|X^{(R)} + Z|_{\mathcal{W}}^2)[B(X^{(R)} + Z, X^{(R)} + Z) + G(X^{(R)} + Z)] = 0, \quad (5.37)$$

with $X^{(R)}(0) = y$. Moreover, define $U^{(R)}(t) = X^{(R)}(t) + Z(t)$, which is a weak solution to equation (5.34). We denote its law on Ω by $P_y^{(R)}$. For the noise, by **Hypothesis H1**, the trajectories of the noise belong to

$$\Omega^* := \bigcap_{\beta \in (0, \frac{1}{2}), \alpha \in [0, \frac{1}{2} + \varepsilon_0)} C^\beta([0, \infty); D(A^\alpha))$$

with probability one. Hence, the analyticity of the semigroup generated by A implies that for each $\omega \in \Omega^*$,

$$Z(\omega) \in C([0, \infty); D(A^{1+\varepsilon_0-\varepsilon})) \subseteq C([0, \infty); \mathcal{W}), \quad (5.38)$$

for every $\varepsilon \in (0, \frac{1}{4})$.

Now, fix $\omega \in \Omega^*$, we will prove that equation (5.37) has a unique global weak solution in $C([0, \infty); \mathcal{W})$.

Denoting by $X = (\kappa^{(R)}, g^{(R)})$, $Z = (Z_1, Z_2)$, $\kappa^{(R)} = v^{(R)} - Z_1$, $g^{(R)} = T - Z_2$, then (5.37) can be rewritten as

$$\begin{aligned} \frac{\partial \kappa^{(R)}}{\partial t} + ((\kappa^{(R)} + Z_1) \cdot \nabla_H)(\kappa^{(R)} + Z_1) + \Phi(\kappa^{(R)} + Z_1) \frac{\partial(\kappa^{(R)} + Z_1)}{\partial z} + f\kappa \times (\kappa^{(R)} + Z_1) \\ + \nabla_H p_b - \int_{-1}^z \nabla_H(g^{(R)} + Z_2)dz' - \Delta \kappa^{(R)} = 0, \end{aligned} \quad (5.39)$$

$$\frac{\partial g^{(R)}}{\partial t} + ((\kappa^{(R)} + Z_1) \cdot \nabla_H)(g^{(R)} + Z_2) + \Phi(\kappa^{(R)} + Z_1) \frac{\partial(g^{(R)} + Z_2)}{\partial z} + \Phi(\kappa^{(R)} + Z_1) - \Delta g^{(R)} = 0. \quad (5.40)$$

(Existence of weak solution) Multiplying (5.39) by $-\Lambda^{3+4\varepsilon_0} \kappa^{(R)}$, integrating over \mathbb{T}^3 , it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 \\ = & \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [((\kappa^{(R)} + Z_1) \cdot \nabla_H)(\kappa^{(R)} + Z_1)] dx dy dz \\ & + \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} \left[\Phi(\kappa^{(R)} + Z_1) \frac{\partial(\kappa^{(R)} + Z_1)}{\partial z} \right] dx dy dz \\ & + \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} \left[f \kappa \times (\kappa^{(R)} + Z_1) + \nabla_H p_b - \int_{-1}^z \nabla_H (g^{(R)} + Z_2) dz' \right] dx dy dz \\ := & \chi_R(|U^{(R)}|_{\mathcal{W}}^2) (I_1 + I_2 + I_3). \end{aligned}$$

For I_1 , we have

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [(\kappa^{(R)} \cdot \nabla_H) \kappa^{(R)}] dx dy dz \right| \\ = & \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [(\kappa^{(R)} \cdot \nabla_H) \kappa^{(R)}] dx dy dz \right| \\ \leq & C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} \kappa^{(R)}| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} \kappa^{(R)}| \\ \leq & C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| \\ \leq & \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^4, \end{aligned}$$

where the first equality follows from Lemma 5.1. In the first inequality, $\sigma_1 + \sigma_2 = \frac{3}{2}$, $s_1 + s_2 = \frac{3}{2}$, we choose

$$\sigma_1 = 1 - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = \frac{3}{2}, \quad s_2 = 0.$$

The second inequality follows from Lemma 5.3. The Young inequality is used in the last inequality. By the same argument, we have

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [(\kappa^{(R)} \cdot \nabla_H) Z_1] dx dy dz \right| \\ = & \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [(\kappa^{(R)} \cdot \nabla_H) Z_1] dx dy dz \right| \\ \leq & C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} Z_1| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} Z_1| \\ \leq & C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| \left(|\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}} Z_1| + |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^{\frac{3}{2}} \kappa^{(R)}| \right) \\ \leq & \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^{\frac{3}{2}} Z_1|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 |\Lambda^{\frac{3}{2}} \kappa^{(R)}|^2, \end{aligned}$$

where $\sigma_1 + \sigma_2 = \frac{3}{2}$, $s_1 + s_2 = \frac{3}{2}$, we choose

$$\sigma_1 = 1, \quad \sigma_2 = \frac{1}{2}, \quad s_1 = \frac{3}{2}, \quad s_2 = 0.$$

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [(Z_1 \cdot \nabla_H) \kappa^{(R)}] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [(Z_1 \cdot \nabla_H) \kappa^{(R)}] dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} Z_1| |\Lambda^{1+\sigma_2} \kappa^{(R)}| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{s_1} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} \kappa^{(R)}| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}} Z_1|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = \frac{3}{2}$, $s_1 + s_2 = \frac{3}{2}$, we choose

$$\sigma_1 = 1 - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = \frac{3}{2}, \quad s_2 = 0.$$

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [(Z_1 \cdot \nabla_H) Z_1] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [(Z_1 \cdot \nabla_H) Z_1] dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} Z_1| |\Lambda^{1+\sigma_2} Z_1| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{s_1} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} Z_1| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^{\frac{3}{2}} Z_1| \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}} Z_1|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = \frac{3}{2}$, $s_1 + s_2 = \frac{3}{2}$, we choose

$$\sigma_1 = 1, \quad \sigma_2 = \frac{1}{2}, \quad s_1 = \frac{3}{2}, \quad s_2 = 0.$$

For I_2 , we have

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [\Phi(\kappa^{(R)}) \frac{\partial \kappa^{(R)}}{\partial z}] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\Phi(\kappa^{(R)}) \frac{\partial \kappa^{(R)}}{\partial z}] dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} \kappa^{(R)}| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{1+s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} \kappa^{(R)}| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^2 \kappa^{(R)}| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{3}{2}-2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{1}{2}+2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{2+\frac{4}{1+4\varepsilon_0}},
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, we choose

$$\sigma_1 = s_2 = \frac{1}{2} - 2\varepsilon_0, \quad \sigma_2 = s_1 = \frac{1}{2} + 2\varepsilon_0.$$

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [\Phi(\kappa^{(R)}) \frac{\partial Z_1}{\partial z}] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\Phi(\kappa^{(R)}) \frac{\partial Z_1}{\partial z}] dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} Z_1| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{1+s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} Z_1| \\
&\leq |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^2 \kappa^{(R)}| \\
&\leq |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{3}{2}-2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{1}{2}+2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^{\frac{4}{1+4\varepsilon_0}},
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, we choose

$$\sigma_1 = \frac{1}{2} - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = 1, \quad s_2 = 0.$$

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [\Phi(Z_1) \frac{\partial \kappa^{(R)}}{\partial z}] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\Phi(Z_1) \frac{\partial \kappa^{(R)}}{\partial z}] dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} Z_1| |\Lambda^{1+\sigma_2} \kappa^{(R)}| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{1+s_1} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} \kappa^{(R)}| \\
&\leq |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^2 \kappa^{(R)}| \\
&\leq |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{3}{2}-2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{1}{2}+2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^{\frac{4}{1+4\varepsilon_0}},
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, we choose

$$\sigma_1 = 0, \quad \sigma_2 = 1, \quad s_1 = \frac{1}{2} + 2\varepsilon_0, \quad s_2 = \frac{1}{2} - 2\varepsilon_0.$$

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [\Phi(Z_1) \frac{\partial Z_1}{\partial z}] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\Phi(Z_1) \frac{\partial Z_1}{\partial z}] dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} Z_1| |\Lambda^{1+\sigma_2} Z_1| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{1+s_1} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} Z_1| \\
&\leq |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^2 Z_1| \\
&\leq \varepsilon |\Lambda^{2+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 |\Lambda^2 Z_1|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, we choose

$$\sigma_1 = 0, \quad \sigma_2 = 1, \quad s_1 = 1, \quad s_2 = 0.$$

For I_3 ,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} \kappa^{(R)} [fk \times (\kappa^{(R)} + Z_1) + \nabla_H p_b - \int_{-1}^z \nabla_H (g^{(R)} + Z_2) dz'] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [fk \times (\kappa^{(R)} + Z_1) + \nabla_H p_b - \int_{-1}^z \nabla_H (g^{(R)} + Z_2) dz'] dx dy dz \right| \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} Z_1|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2,
\end{aligned}$$

thus,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C(R + |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2)^{2+\frac{4}{1+4\varepsilon_0}} + C(R + |\Lambda^2 Z_1|^2) |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 + |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2.
\end{aligned} \tag{5.41}$$

Multiplying (5.40) by $-\Lambda^{3+4\varepsilon_0} g^{(R)}$, integrating over \mathbb{T}^3 , it follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 + |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 \\
&= \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [((\kappa^{(R)} + Z_1) \cdot \nabla_H)(g^{(R)} + Z_2)] dx dy dz \\
&+ \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [\Phi(\kappa^{(R)} + Z_1) \frac{\partial(g^{(R)} + Z_2)}{\partial z}] dx dy dz \\
&+ \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} \Phi(\kappa^{(R)} + Z_1) dx dy dz \\
&:= \chi_R(|U^{(R)}|_{\mathcal{W}}^2) (I_4 + I_5 + I_6).
\end{aligned}$$

For I_4 , we have

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [(\kappa^{(R)} \cdot \nabla_H) g^{(R)}] dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)} \Lambda^{\frac{1}{2}+2\varepsilon_0} [(\kappa^{(R)} \cdot \nabla_H) g^{(R)}] dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} g^{(R)}| + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}| |\Lambda^{s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} g^{(R)}| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}| |\Lambda^{\frac{3}{2}} \kappa^{(R)}| \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}} \kappa^{(R)}|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = \frac{3}{2}$, $s_1 + s_2 = \frac{3}{2}$, we choose

$$\sigma_1 = 1 - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = \frac{3}{2}, \quad s_2 = 0.$$

By the same argument, we have

$$\begin{aligned}
\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [(\kappa^{(R)} \cdot \nabla_H) Z_2] dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^{\frac{3}{2}} Z_2|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2 |\Lambda^{\frac{3}{2}} \kappa^{(R)}|^2, \\
\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [Z_1 \cdot \nabla_H g^{(R)}] dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}} Z_1|^2, \\
\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [Z_1 \cdot \nabla_H Z_2] dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2 |\Lambda^{\frac{3}{2}} Z_1|^2.
\end{aligned}$$

For I_5 , similar to I_4 ,

$$\begin{aligned}
\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [\Phi(\kappa^{(R)}) \frac{\partial g^{(R)}}{\partial z}] dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{4}{1+4\varepsilon_0}}, \\
\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [\Phi(\kappa^{(R)}) \frac{\partial Z_2}{\partial z}] dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^2 Z_2|^2, \\
\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [\Phi(Z_1) \frac{\partial g^{(R)}}{\partial z}] dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^{\frac{4}{1+4\varepsilon_0}}, \\
\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} [\Phi(Z_1) \frac{\partial Z_2}{\partial z}] dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 |\Lambda^2 Z_2|^2.
\end{aligned}$$

For I_6 , we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}^3} \Lambda^{3+4\varepsilon_0} g^{(R)} \Phi(\kappa^{(R)} + Z_1) dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} (\kappa^{(R)} + Z_1)| \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 + |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C(R + |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^{\frac{4}{1+4\varepsilon_0}} |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2 + |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 |\Lambda^2 Z_2|^2).
\end{aligned} \tag{5.42}$$

Since $|X^{(R)}|_{\mathcal{W}}^2 = |\kappa^{(R)}|_{\mathcal{W}}^2 + |g^{(R)}|_{\mathcal{W}}^2$, combining (5.41) and (5.42), we have

$$\frac{d|X^{(R)}|_{\mathcal{W}}^2}{dt} + \|X^{(R)}\|_{\frac{5}{2}+2\varepsilon_0}^2 \leq C(R + |\Lambda^2 Z|^2)^{2+\frac{4}{1+4\varepsilon_0}}, \tag{5.43}$$

by the property of Z in (5.38), (5.37) has a weak solution $X^{(R)}$ in $L^\infty([0, T]; \mathcal{W}) \cap L^2([0, T]; D(A^{\frac{5}{4}+\varepsilon_0}))$.

(Continuity of weak solution) Multiplying (5.39) by $-\Lambda^{1+4\varepsilon_0} \frac{d\kappa^{(R)}}{dt}$, integrating over \mathbb{T}^3 , it follows that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 dx dy dz + |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 \\
&= \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} [((\kappa^{(R)} + Z_1) \cdot \nabla_H)(\kappa^{(R)} + Z_1)] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \\
&+ \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \left[\Phi(\kappa^{(R)} + Z_1) \frac{\partial(\kappa^{(R)} + Z_1)}{\partial z} \right] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \\
&+ \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} (fk \times (\kappa^{(R)} + Z_1) + \nabla_H p_b - \int_{-1}^z \nabla_H(g^{(R)} + Z_2) dz') \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \\
&:= \chi_R(|U^{(R)}|_{\mathcal{W}}^2)(J_1 + J_2 + J_3),
\end{aligned}$$

where $\dot{\kappa}^{(R)}$ denotes $\frac{d\kappa^{(R)}}{dt}$, and this symbol always denotes the deviation with respect to t .

For J_1 , By Hölder inequality and the Young inequality, we have

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [(\kappa^{(R)} \cdot \nabla_H) \kappa^{(R)}] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} \kappa^{(R)}| + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} \kappa^{(R)}| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}} \kappa^{(R)}| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^4,
\end{aligned}$$

similarly,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [(\kappa^{(R)} \cdot \nabla_H) Z_1] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} Z_1| + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} Z_1| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| (|\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^{\frac{3}{2}} Z_1| + |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^{\frac{3}{2}} \kappa^{(R)}|) \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2,
\end{aligned}$$

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [(Z_1 \cdot \nabla_H) \kappa^{(R)}] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} Z_1| |\Lambda^{1+\sigma_2} \kappa^{(R)}| + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{s_1} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} \kappa^{(R)}| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| (|\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^{\frac{3}{2}} \kappa^{(R)}| + |\Lambda^{\frac{3}{2}} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|) \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 |\Lambda^{\frac{3}{2}} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}} Z_1|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2,
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [(Z_1 \cdot \nabla_H) Z_1] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} Z_1| |\Lambda^{1+\sigma_2} Z_1| + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{s_1} Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} Z_1| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^{\frac{3}{2}} Z_1| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 |\Lambda^{\frac{3}{2}} Z_1|^2.
\end{aligned}$$

For J_2 , we obtain

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [\Phi(\kappa^{(R)}) \frac{\partial \kappa^{(R)}}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| (|\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} \kappa^{(R)}| + |\Lambda^{1+s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} \kappa^{(R)}|) \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^2 \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{1}{2}+2\varepsilon_0} |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{1}{2}-2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{4}{1+4\varepsilon_0}},
\end{aligned}$$

similarly,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [\Phi(\kappa^{(R)}) \frac{\partial Z_1}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| \left(|\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} Z_1| + |\Lambda^{1+s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} Z_1| \right) \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^2 Z_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^2 Z_1|^2, \\
& \left| \int_{\mathbb{T}^3} [\Phi(Z_1) \frac{\partial \kappa^{(R)}}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}| |\Lambda^2 Z_1| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^2 Z_1|^2, \\
& \left| \int_{\mathbb{T}^3} [\Phi(Z_1) \frac{\partial Z_1}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1| |\Lambda^2 Z_1| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2 |\Lambda^2 Z_1|^2.
\end{aligned}$$

For J_3 , we have

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [fk \times (\kappa^{(R)} + Z_1) + \nabla_H p b - \int_{-1}^z \nabla_H (g^{(R)} + Z_2) dz'] \Lambda^{1+4\varepsilon_0} \dot{\kappa}^{(R)} dx dy dz \right| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{\kappa}^{(R)}|^2 + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} (\kappa^{(R)} + Z_1)|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} (g^{(R)} + Z_2)|^2.
\end{aligned}$$

Multiplying (5.40) by $-\Lambda^{1+4\varepsilon_0} \frac{d}{dt} g^{(R)}$, integrating over \mathbb{T}^3 , it follows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 dx dy dz + |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 \\
& = \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} [((\kappa^{(R)} + Z_1) \cdot \nabla_H)(g^{(R)} + Z_2)] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz, \\
& + \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} [\Phi(\kappa^{(R)} + Z_1) \frac{\partial (g^{(R)} + Z_2)}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz, \\
& + \chi_R(|U^{(R)}|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Phi(\kappa^{(R)} + Z_1) \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz, \\
& := \chi_R(|U^{(R)}|_{\mathcal{W}}^2) (J_4 + J_5 + J_6),
\end{aligned}$$

where $\dot{g}^{(R)}$ denotes $\frac{dg^{(R)}}{dt}$.

For J_4 , we have

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [(\kappa^{(R)} \cdot \nabla_H) g^{(R)}] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}| \left(|\Lambda^{\frac{1}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}| |\Lambda^{1+\sigma_2} g^{(R)}| + |\Lambda^{s_1} \kappa^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} g^{(R)}| \right) \\
& \leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}| |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}| |\Lambda^{\frac{3}{2}} \kappa^{(R)}| \\
& \leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}} \kappa^{(R)}|^2,
\end{aligned}$$

similarly,

$$\begin{aligned}
\left| \int_{\mathbb{T}^3} [(\kappa^{(R)} \cdot \nabla_H) Z_2] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2 |\Lambda^{\frac{3}{2}} \kappa^{(R)}|^2, \\
\left| \int_{\mathbb{T}^3} [(Z_1 \cdot \nabla_H) g^{(R)}] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}} Z_1|^2, \\
\left| \int_{\mathbb{T}^3} [(Z_1 \cdot \nabla_H) Z_2] \Lambda^{4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2 |\Lambda^{\frac{3}{2}} Z_1|^2.
\end{aligned}$$

For J_5 , we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}^3} [\Phi(\kappa^{(R)}) \frac{\partial g^{(R)}}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| \\
&\leq |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}| \|\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} \kappa^{(R)}\| |\Lambda^{1+\sigma_2} g^{(R)}|^2 + C |\Lambda^{2\varepsilon_0} \dot{g}^{(R)}| \|\Lambda^{1+s_1} \kappa^{(R)}\| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} g^{(R)}|^2 \\
&\leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}| \|\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}\| |\Lambda^2 g^{(R)}|^2 \\
&\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^{\frac{4}{1+4\varepsilon_0}},
\end{aligned}$$

similarly, we get

$$\begin{aligned}
\left| \int_{\mathbb{T}^3} [\Phi(\kappa^{(R)}) \frac{\partial Z_2}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^{\frac{4}{1+4\varepsilon_0}}, \\
\left| \int_{\mathbb{T}^3} [\Phi(Z_1) \frac{\partial Z_2}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g^{(R)}|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^{\frac{4}{1+4\varepsilon_0}}, \\
\left| \int_{\mathbb{T}^3} [\Phi(Z_1) \frac{\partial Z_2}{\partial z}] \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| &\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^2 Z_1|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_2|^2.
\end{aligned}$$

For J_6 , we have

$$\begin{aligned}
&\left| \int_{\mathbb{T}^3} \Phi(\kappa^{(R)} + Z_1) \Lambda^{1+4\varepsilon_0} \dot{g}^{(R)} dx dy dz \right| \\
&\leq C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}| \|\Lambda^{\frac{3}{2}+2\varepsilon_0} (\kappa^{(R)} + Z_1)\| \\
&\leq \varepsilon |\Lambda^{\frac{1}{2}+2\varepsilon_0} \dot{g}^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} Z_1|^2.
\end{aligned}$$

Combing the above estimations, we obtain

$$\begin{aligned}
&\frac{d}{dt} \|X^{(R)}\|_{\frac{3}{2}+2\varepsilon_0}^2 + \|\dot{X}^{(R)}\|_{\frac{1}{2}+2\varepsilon_0}^2 \\
&\leq C_{\mathcal{X}\mathcal{R}}(|U^{(R)}|_{\mathcal{W}}^2) \left(C(R) + |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 + |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 + |\Lambda^2 Z_1|^{2+\frac{4}{1+4\varepsilon_0}} + |\Lambda^2 Z_2|^{2+\frac{4}{1+4\varepsilon_0}} \right).
\end{aligned} \tag{5.44}$$

Integrating (5.44) on t from 0 to T , as $\int_0^T |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa^{(R)}|^2 dt$ and $\int_0^T |\Lambda^{\frac{5}{2}+2\varepsilon_0} g^{(R)}|^2 dt$ can be dominated by (5.43), by the property of Z in (5.38), we get the time derivative $\frac{dX^{(R)}}{dt} \in L^2([0, T]; D(A^{\frac{1}{4}+\varepsilon_0}))$. Then by (5.43) and [25], we obtain $X^{(R)} \in C([0, T]; \mathcal{W})$.

(Uniqueness of weak solution) Let $X_1 = (\kappa_1^{(R)}, g_1^{(R)})$, $X_2 = (\kappa_2^{(R)}, g_2^{(R)})$ be two solutions of (5.37) in $C([0, T]; \mathcal{W})$ and set

$$\begin{aligned}
Y &= X_1 - X_2 = (\kappa, g) = (\kappa_1^{(R)} - \kappa_2^{(R)}, g_1^{(R)} - g_2^{(R)}), \\
v_1 &= \kappa_1^{(R)} + Z_1, \quad T_1 = g_1^{(R)} + Z_2, \quad v_2 = \kappa_2^{(R)} + Z_1, \quad T_2 = g_2^{(R)} + Z_2.
\end{aligned}$$

Firstly, from (5.37), we obtain

$$\begin{aligned} \frac{dY}{dt} + AY &+ B(U_1, U_1)(\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) + B(U_1, Y)\chi_R(|U_2|_{\mathcal{W}}^2) + B(Y, U_2)\chi_R(|U_2|_{\mathcal{W}}^2) \\ &+ G(U_1)(\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) + \tilde{G}(Y)\chi_R(|U_2|_{\mathcal{W}}^2) = 0, \end{aligned}$$

where

$$\tilde{G}(Y) = \begin{pmatrix} fk \times \kappa - \int_{-1}^z \nabla_H g dz' \\ \Phi(\kappa) \end{pmatrix}.$$

That is,

$$\begin{aligned} \frac{d\kappa}{dt} &+ A_1\kappa + \left((v_1 \cdot \nabla_H)v_1 + \Phi(v_1)\frac{\partial v_1}{\partial z} \right) (\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) \\ &+ \left((v_1 \cdot \nabla_H)\kappa + \Phi(v_1)\frac{\partial \kappa}{\partial z} \right) \chi_R(|U_2|_{\mathcal{W}}^2) + \left((\kappa \cdot \nabla_H)v_2 + \Phi(\kappa)\frac{\partial v_2}{\partial z} \right) \chi_R(|U_2|_{\mathcal{W}}^2) \\ &+ (fk \times v_1 + \nabla_H p_b - \int_{-1}^z \nabla_H T_1 dz') (\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) \\ &+ (fk \times \kappa - \int_{-1}^z \nabla_H g dz') \chi_R(|U_2|_{\mathcal{W}}^2) = 0, \end{aligned} \quad (5.45)$$

and

$$\begin{aligned} \frac{dg}{dt} &+ A_2g + \left((v_1 \cdot \nabla_H)T_1 + \Phi(v_1)\frac{\partial T_1}{\partial z} \right) (\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) \\ &+ \left((v_1 \cdot \nabla_H)g + \Phi(v_1)\frac{\partial g}{\partial z} \right) \chi_R(|U_2|_{\mathcal{W}}^2) + \left((\kappa \cdot \nabla_H)T_2 + \Phi(\kappa)\frac{\partial T_2}{\partial z} \right) \chi_R(|U_2|_{\mathcal{W}}^2) \\ &+ \Phi(\kappa)\chi_R(|U_2|_{\mathcal{W}}^2) = 0. \end{aligned} \quad (5.46)$$

Multiplying (5.45) by $-\Lambda^{1+4\varepsilon_0}\kappa$, then integrating over \mathbb{T}^3 , we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\Lambda^{\frac{1}{2}+2\varepsilon_0}\kappa|^2 dx dy dz + |\Lambda^{\frac{3}{2}+2\varepsilon_0}\kappa|^2 \\ &= (\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) \int_{\mathbb{T}^3} [(v_1 \cdot \nabla_H)v_1 + \Phi(v_1)\frac{\partial v_1}{\partial z}] \Lambda^{1+4\varepsilon_0}\kappa dx dy dz \\ &+ \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} [(v_1 \cdot \nabla_H)\kappa + \Phi(v_1)\frac{\partial \kappa}{\partial z}] \Lambda^{1+4\varepsilon_0}\kappa dx dy dz \\ &+ \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} [(\kappa \cdot \nabla_H)v_2 + \Phi(\kappa)\frac{\partial v_2}{\partial z}] \Lambda^{1+4\varepsilon_0}\kappa dx dy dz \\ &+ (\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) \int_{\mathbb{T}^3} (fk \times v_1 + \nabla_H p_b - \int_{-1}^z \nabla_H T_1 dz') \Lambda^{1+4\varepsilon_0}\kappa dx dy dz \\ &+ \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} (fk \times \kappa - \int_{-1}^z \nabla_H g dz') \Lambda^{1+4\varepsilon_0}\kappa dx dy dz \\ &:= I + II + III + IV + V. \end{aligned}$$

It's easy to know that

$$|\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)| \leq C(R)|Y|_{\mathcal{W}}[I_{[0,R+1]}(|U_1|_{\mathcal{W}}^2) + I_{[0,R+1]}(|U_2|_{\mathcal{W}}^2)].$$

For I , since $\int_{\mathbb{T}^3} ((v_1 \cdot \nabla_H) v_1) \Lambda^{1+4\varepsilon_0} \kappa dx dy dz$ is weaker than $\int_{\mathbb{T}^3} (\Phi(v_1) \frac{\partial v_1}{\partial z}) \Lambda^{1+4\varepsilon_0} \kappa dx dy dz$, we only need to estimate the term involved Φ . For $\varepsilon_1 \in (0, 2\varepsilon_0)$, we have

$$\begin{aligned}
& C(R) |Y|_{\mathcal{W}} \left| \int_{\mathbb{T}^3} [\Phi(v_1) \frac{\partial v_1}{\partial z}] \Lambda^{1+4\varepsilon_0} \kappa dx dy dz \right| \\
& \leq C(R) |Y|_{\mathcal{W}} \left| \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0 - \frac{1}{2} + \varepsilon_1} [\Phi(v_1) \frac{\partial v_1}{\partial z}] \Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa dx dy dz \right| \\
& \leq C(R) |Y|_{\mathcal{W}} \left(|\Lambda^{2\varepsilon_0 + \frac{1}{2} + \varepsilon_1 + \sigma_1} v_1| |\Lambda^{1+\sigma_2} v_1| |\Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa| + |\Lambda^{1+s_1} v_1| |\Lambda^{2\varepsilon_0 + \frac{1}{2} + \varepsilon_1 + s_2} v_1| |\Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa| \right) \\
& \leq C(R) |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda^{1+\varepsilon_1} v_1| |\Lambda^{\frac{3}{2} + 2\varepsilon_0} v_1| |\Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa| + |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g| |\Lambda^{1+\varepsilon_1} v_1| |\Lambda^{\frac{3}{2} + 2\varepsilon_0} v_1| |\Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa| \\
& \leq C(R) \left(|\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| + |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g| \right) |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^{\varepsilon_1} |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^{1-\varepsilon_1} |\Lambda^{\frac{3}{2} + 2\varepsilon_0} v_1|^2 \\
& \leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g|^2 + C(R) |\Lambda^{\frac{3}{2} + 2\varepsilon_0} v_1|^{\frac{4}{\varepsilon_1}} |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 \\
& \leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g|^2 + C(R, |U_1|_{\mathcal{W}}^2) |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = s_2 = \frac{1}{2} - 2\varepsilon_0, \quad \sigma_2 = s_1 = \frac{1}{2} + 2\varepsilon_0.$$

For II ,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [\Phi(v_1) \frac{\partial \kappa}{\partial z}] \Lambda^{1+4\varepsilon_0} \kappa dx dy dz \right| \\
& \leq \left| \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0 - \frac{1}{2}} [\Phi(v_1) \frac{\partial \kappa}{\partial z}] \Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa dx dy dz \right| \\
& \leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda^{2\varepsilon_0 + \frac{1}{2} + \sigma_1} v_1| |\Lambda^{1+\sigma_2} \kappa| + C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda^{1+s_1} v_1| |\Lambda^{2\varepsilon_0 + \frac{1}{2} + s_2} \kappa| \\
& \leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda^{\frac{3}{2} + 2\varepsilon_0} v_1| |\Lambda \kappa| \\
& \leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda^{\frac{3}{2} + \varepsilon_0} v_1| |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^{\frac{1}{2} + 2\varepsilon_0} |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^{\frac{1}{2} - 2\varepsilon_0} \\
& \leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} v_1|^{\frac{4}{1+4\varepsilon_0}} |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 \\
& \leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C(R, |U_1|_{\mathcal{W}}^2) |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = 1, \quad \sigma_2 = 0, \quad s_1 = \frac{1}{2} + 2\varepsilon_0, \quad s_2 = \frac{1}{2} - 2\varepsilon_0.$$

For III ,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} (\Phi(\kappa) \frac{\partial v_2}{\partial z}) \Lambda^{1+4\varepsilon_0} \kappa dx dy dz \right| \\
& \leq \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0 - \frac{1}{2}} [\Phi(\kappa) \frac{\partial v_2}{\partial z}] \Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa dx dy dz \\
& \leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda^{2\varepsilon_0 + \frac{1}{2} + \sigma_1} \kappa| |\Lambda^{1+\sigma_2} v_2| + C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda^{1+s_1} \kappa| |\Lambda^{2\varepsilon_0 + \frac{1}{2} + s_2} v_2| \\
& \leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa| |\Lambda \kappa| |\Lambda^{\frac{3}{2} + 2\varepsilon_0} v_2| \\
& \leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C(R, |U_2|_{\mathcal{W}}^2) |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = \frac{1}{2} - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = 0, \quad s_2 = 1.$$

For IV ,

$$\begin{aligned} & \left| (\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2)) \int_{\mathbb{T}^3} [fk \times v_1 + \nabla_H p_b - \int_{-1}^z \nabla_H T_1 dz'] \Lambda^{1+4\varepsilon_0} \kappa dx dy dz \right| \\ & \leq C(R) [I_{[0,R+1]}(|U_1|_{\mathcal{W}}^2) + I_{[0,R+1]}(|U_2|_{\mathcal{W}}^2)] |Y|_{\mathcal{W}} \left| \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0 - \frac{1}{2} + \varepsilon_1} [fk \times v_1 \right. \\ & \quad \left. + \nabla_H p_b - \int_{-1}^z \nabla_H T_1 dz'] \Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa dx dy dz \right| \\ & \leq C(R) [I_{[0,R+1]}(|U_1|_{\mathcal{W}}^2) + I_{[0,R+1]}(|U_2|_{\mathcal{W}}^2)] |Y|_{\mathcal{W}} \Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa |\Lambda^{2\varepsilon_0 - \frac{1}{2} + \varepsilon_1} v_1| \\ & \quad + C(R) [I_{[0,R+1]}(|U_1|_{\mathcal{W}}^2) + I_{[0,R+1]}(|U_2|_{\mathcal{W}}^2)] |Y|_{\mathcal{W}} \Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa |\Lambda^{2\varepsilon_0 + \frac{1}{2} + \varepsilon_1} T_1| \\ & := C(R) [I_{[0,R+1]}(|U_1|_{\mathcal{W}}^2) + I_{[0,R+1]}(|U_2|_{\mathcal{W}}^2)] (IV^{(1)} + IV^{(2)}), \end{aligned}$$

for $IV^{(1)}$, we have

$$\begin{aligned} IV^{(1)} &= |Y|_{\mathcal{W}} \Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa |\Lambda^{2\varepsilon_0 - \frac{1}{2} + \varepsilon_1} v_1| \\ &\leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^{2-\varepsilon_1} |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^{\varepsilon_1} |\Lambda^{2\varepsilon_0 - \frac{1}{2} + \varepsilon_1} v_1| + |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g| |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^{\varepsilon_1} |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^{1-\varepsilon_1} |\Lambda^{2\varepsilon_0 - \frac{1}{2} + \varepsilon_1} v_1| \\ &\leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 |\Lambda^{2\varepsilon_0 - \frac{1}{2} + \varepsilon_1} v_1|^{\frac{2}{\varepsilon_1}} \\ &\leq \varepsilon |\Lambda^{1+2\varepsilon_0} \kappa|^2 + \varepsilon |\Lambda^{1+2\varepsilon_0} \kappa|^2 + C(R, |U_1|_{\mathcal{W}}^2) |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 \end{aligned}$$

$IV^{(2)}$ is similar to $IV^{(1)}$, we have

$$\begin{aligned} IV^{(2)} &= |Y|_{\mathcal{W}} \Lambda^{\frac{3}{2} + 2\varepsilon_0 - \varepsilon_1} \kappa |\Lambda^{\frac{1}{2} + 2\varepsilon_0 + \varepsilon_1} T_1| \\ &\leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 |\Lambda^{2\varepsilon_0 + \frac{1}{2} + \varepsilon_1} T_1|^{\frac{2}{\varepsilon_1}} \\ &\leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C(R, |U_1|_{\mathcal{W}}^2) |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2. \end{aligned}$$

For V , we have

$$\begin{aligned} & \left| \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} (fk \times \kappa - \int_{-1}^z \nabla_H g dz') \Lambda^{1+4\varepsilon_0} \kappa dx dy dz \right| \\ & \leq \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0 - \frac{1}{2}} [fk \times \kappa - \int_{-1}^z \nabla_H g dz'] \Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa dx dy dz \\ & \leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C |\Lambda^{2\varepsilon_0 - \frac{1}{2}} \kappa|^2 + C |\Lambda^{\frac{1}{2} + 2\varepsilon_0} g|^2, \end{aligned}$$

thus, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 dx dy dz + |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 \\ & \leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C(R, |U_1|_{\mathcal{W}}, |U_2|_{\mathcal{W}}) (1 + |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 + |\Lambda^{\frac{1}{2} + 2\varepsilon_0} g|^2). \end{aligned} \tag{5.47}$$

Multiplying (5.46) by $-\Lambda^{1+4\varepsilon_0}g$, then integrating over \mathbb{T}^3 , we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\Lambda^{\frac{1}{2}+2\varepsilon_0} g|^2 dx dy dz + |\Lambda^{\frac{3}{2}+2\varepsilon_0} g|^2 \\
&= \left(\chi_R(|U_1|_{\mathcal{W}}^2) - \chi_R(|U_2|_{\mathcal{W}}^2) \right) \int_{\mathbb{T}^3} \left[(v_1 \cdot \nabla_H) T_1 + \Phi(v_1) \frac{\partial T_1}{\partial z} \right] \Lambda^{1+4\varepsilon_0} g dx dy dz \\
&+ \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \left[(v_1 \cdot \nabla_H) g + \Phi(v_1) \frac{\partial g}{\partial z} \right] \Lambda^{1+4\varepsilon_0} g dx dy dz \\
&+ \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \left[(\kappa \cdot \nabla_H) T_2 + \Phi(\kappa) \frac{\partial T_2}{\partial z} \right] \Lambda^{1+4\varepsilon_0} g dx dy dz \\
&+ \chi_R(|U_2|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \Phi(\kappa) \Lambda^{1+4\varepsilon_0} g dx dy dz \\
&:= VI + VII + VIII + IX.
\end{aligned}$$

For VI, similar to I, we only need to estimate

$$\begin{aligned}
& |Y|_{\mathcal{W}} \left| \int_{\mathbb{T}^3} \left[\Phi(v_1) \frac{\partial T_1}{\partial z} \right] \Lambda^{1+4\varepsilon_0} g dx dy dz \right| \\
&= |Y|_{\mathcal{W}} \left| \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0-\frac{1}{2}+\varepsilon_1} \left[\Phi(v_1) \frac{\partial T_1}{\partial z} \right] \Lambda^{\frac{3}{2}+2\varepsilon_0-\varepsilon_1} g dx dy dz \right| \\
&\leq |Y|_{\mathcal{W}} |\Lambda^{\frac{3}{2}+2\varepsilon_0-\varepsilon_1} g| \left(|\Lambda^{\frac{1}{2}+2\varepsilon_0+\varepsilon_1+\sigma_1} v_1| |\Lambda^{1+\sigma_2} T_1| + |\Lambda^{1+s_1} v_1| |\Lambda^{\frac{1}{2}+2\varepsilon_0+\varepsilon_1+s_2} T_1| \right) \\
&\leq |Y|_{\mathcal{W}} |\Lambda^{\frac{3}{2}+2\varepsilon_0-\varepsilon_1} g| |\Lambda^{1+\varepsilon_1} v_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T_1| \\
&\leq C \left(|\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa| + |\Lambda^{\frac{3}{2}+2\varepsilon_0} g| \right) |\Lambda^{\frac{1}{2}+2\varepsilon_0} g|^{\varepsilon_1} |\Lambda^{\frac{3}{2}+2\varepsilon_0} g|^{1-\varepsilon_1} |\Lambda^{1+\varepsilon_1} v_1| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T_1| \\
&\leq \varepsilon |\Lambda^{\frac{3}{2}+2\varepsilon_0} g|^2 + \varepsilon |\Lambda^{\frac{3}{2}+2\varepsilon_0} \kappa|^2 + C |\Lambda^{1+2\varepsilon_0} v_1|^{\frac{2}{\varepsilon_1}} |\Lambda^{\frac{3}{2}+2\varepsilon_0} T_1|^{\frac{2}{\varepsilon_1}} |\Lambda^{\frac{1}{2}+2\varepsilon_0} g|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = \frac{1}{2} - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = \varepsilon_1, \quad s_2 = 1 - \varepsilon_1.$$

For VII,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \left[\Phi(v_1) \frac{\partial g}{\partial z} \right] \Lambda^{1+4\varepsilon_0} g dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0-\frac{1}{2}} \left[\Phi(v_1) \frac{\partial g}{\partial z} \right] \Lambda^{\frac{3}{2}+2\varepsilon_0} g dx dy dz \right| \\
&\leq C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g| \left(|\Lambda^{2\varepsilon_0+\frac{1}{2}+\sigma_1} v_1| |\Lambda^{1+\sigma_2} g| + |\Lambda^{1+s_1} v_1| |\Lambda^{2\varepsilon_0+\frac{1}{2}+s_2} g| \right) \\
&\leq C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g| |\Lambda^1 g| |\Lambda^{\frac{3}{2}+2\varepsilon_0} v_1| \\
&\leq C |\Lambda^{\frac{3}{2}+2\varepsilon_0} g| |\Lambda^{\frac{1}{2}+2\varepsilon_0} g|^{\frac{1}{2}+2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} g|^{\frac{1}{2}-2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} v_1| \\
&\leq \varepsilon |\Lambda^{\frac{3}{2}+2\varepsilon_0} g|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v_1|^{\frac{4}{1+4\varepsilon_0}} |\Lambda^{\frac{1}{2}+2\varepsilon_0} g|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = 1, \quad \sigma_2 = 0, \quad s_1 = \frac{1}{2} + 2\varepsilon_0, \quad s_2 = \frac{1}{2} - 2\varepsilon_0.$$

For VIII,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [\Phi(\kappa) \frac{\partial T_2}{\partial z}] \Lambda^{1+4\varepsilon_0} g dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0 - \frac{1}{2}} [\Phi(\kappa) \frac{\partial T_2}{\partial z}] \Lambda^{\frac{3}{2} + 2\varepsilon_0} g dx dy dz \right| \\
&\leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g| (|\Lambda^{\frac{1}{2} + 2\varepsilon_0 + \sigma_1} \kappa| |\Lambda^{1 + \sigma_2} T_2| + |\Lambda^{1 + s_1} \kappa| |\Lambda^{\frac{1}{2} + 2\varepsilon_0 + s_2} T_2|) \\
&\leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g| |\Lambda^1 \kappa| |\Lambda^{\frac{3}{2} + 2\varepsilon_0} T_2| \\
&\leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g| |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^{\frac{1}{2} + 2\varepsilon_0} |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^{\frac{1}{2} - 2\varepsilon_0} |\Lambda^{\frac{3}{2} + 2\varepsilon_0} T_2| \\
&\leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g|^2 + \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \kappa|^2 + C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} T_2|^{\frac{4}{1+4\varepsilon_0}} |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2,
\end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = \frac{1}{2} - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = 0, \quad s_2 = 1.$$

For IX, we have

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \Phi(\kappa) \Lambda^{1+4\varepsilon_0} g dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{2\varepsilon_0 - \frac{1}{2}} \Phi(\kappa) \Lambda^{\frac{3}{2} + 2\varepsilon_0} g dx dy dz \right| \\
&\leq C |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g| |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa| \\
&\leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g|^2 + C |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2,
\end{aligned}$$

thus, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\Lambda^{\frac{1}{2} + 2\varepsilon_0} g|^2 dx dy dz + |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g|^2 \\
&\leq \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} g|^2 + \varepsilon |\Lambda^{\frac{3}{2} + 2\varepsilon_0} \eta|^2 + C(R, |U_1|_{\mathcal{W}}, |U_2|_{\mathcal{W}}) (1 + |\Lambda^{\frac{1}{2} + 2\varepsilon_0} \kappa|^2 + |\Lambda^{\frac{1}{2} + 2\varepsilon_0} g|^2).
\end{aligned} \tag{5.48}$$

Combining (5.47) with (5.48), we have

$$\frac{d\|Y\|_{\frac{1}{2} + 2\varepsilon_0}^2}{dt} + \|Y\|_{\frac{3}{2} + 2\varepsilon_0}^2 \leq C(R, \sup_{t \in [0, T]} |U_1|_{\mathcal{W}}, \sup_{t \in [0, T]} |U_2|_{\mathcal{W}}) (1 + \|Y\|_{\frac{1}{2} + 2\varepsilon_0}^2), \tag{5.49}$$

by Gronwall inequality, (5.49) yields that $\|Y\|_{\frac{1}{2} + 2\varepsilon_0} = 0$, which implies $Y = 0$.

Up to now, we have proved that equation (5.37) has a unique global weak solution in the space $C([0, T]; \mathcal{W})$.

Next, we prove (5.35). In order to do so, it is sufficient to show that $P_y^{(R)}[\tau_R < \varepsilon] \leq C(\varepsilon, R)$ with $C(\varepsilon, R) \downarrow 0$ as $\varepsilon \downarrow 0$, for all $y \in \mathcal{W}$, with $|y|_{\mathcal{W}}^2 \leq \frac{R}{8}$. So, fix ε small enough, let

$$\Theta_{\varepsilon, R} := \sup_{t \in [0, \varepsilon]} |AZ(t)|$$

and assume that $\Theta_{\varepsilon,R}^6 \leq \frac{R}{8}$. Moreover, setting

$$\varphi(t) := |X^{(R)}|_{\mathcal{W}}^2 + \Theta_{\varepsilon,R}^6,$$

by (5.43), we get $\dot{\varphi} \leq C(R)$. This implies, together with the bounds on y and $\Theta_{\varepsilon,R}$, that

$$\sup_{t \in [0, \varepsilon]} |U^{(R)}(t)|_{\mathcal{W}}^2 \leq R$$

for ε small enough. It follows that $\tau_R \geq \varepsilon$. Hence,

$$P_y^{(R)}[\tau_R < \varepsilon] \leq P_y^{(R)}\left[\sup_{t \in [0, \varepsilon]} |AZ|^6 > \frac{R}{8}\right],$$

letting $\varepsilon \downarrow 0$, we have

$$P_y^{(R)}[\tau_R < \varepsilon] \rightarrow 0.$$

Since the probability above is independent of y , (5.35) is proved. Finally, since

$$U(t \wedge \tau_R(U^{(R)})) = U^{(R)}(t \wedge \tau_R(U^{(R)})) \quad \forall t, \quad P - a.s.$$

and U is H -valued weakly continuous, we obtain $\tau_R(U^{(R)}) = \tau_R(U)$, thus (5.36) is proved. ■

In order to apply Theorem 5.4 in [8] to obtain Theorem 1.2, we now only need to prove

Proposition 5.2. *Assume **Hypothesis H1** holds. For every $R > 0$, the transition semigroup $(P_t^{(R)})_{t \geq 0}$ associated to equation (5.34) is \mathcal{W} -strong Feller.*

Proof of Proposition 5.2 Let $(\Omega, \mathcal{B}, (\mathcal{B}_t)_{t \geq 0}, P)$ be a filtered probability space, $(W_t)_{t \geq 0}$ a cylindrical Wiener process on H and for every $y \in \mathcal{W}$, denote by $U_y^{(R)}$ the solution to (5.34) with initial value $y \in \mathcal{W}$. By the Bismut-Elworthy-Li formula,

$$D_z(P_t^{(R)}\psi)(y) = \frac{1}{t} E^{\mathbb{P}} \left[\psi(U_y^{(R)}(t)) \int_0^t (Q^{-\frac{1}{2}} D_z Y_y^{(R)}(s), dW(s)) \right],$$

where $D_z(P_t^{(R)}\psi)$ denotes $(D(P_t^{(R)}\psi), y)$, for $y \in H$, $D_z Y_y^{(R)} = DY_y^{(R)} \cdot z$, and $DY_y^{(R)}$ denotes the derivative of $U_y^{(R)}$ with respect to the initial value. Then for $|\psi|_{\infty} \leq 1$, by the B-D-G inequality, we have

$$|(P_t^{(R)}\psi)(y_0 + h) - (P_t^{(R)}\psi)(y_0)| \leq \frac{C}{t} \sup_{\eta \in [0, 1]} E^{\mathbb{P}} \left[\left(\int_0^t |Q^{-\frac{1}{2}} D_h Y_{y_0 + \eta h}^{(R)}(s)| ds \right)^{\frac{1}{2}} \right]. \quad (5.50)$$

This proposition is proved once we prove that the right side of the above inequality converges to 0 as $|h|_{\mathcal{W}} \rightarrow 0$.

For any $y \in \mathcal{W}$, $h \in H$, write $U = U_y^{(R)}$, $DY = D_h U = (D_h v, D_h T) = (\eta(t, y) \cdot h, \gamma(t, y) \cdot h)$ and denote $DY = \beta = (\eta, \gamma)$ for simplicity. Refer to (5.34) and [6], we have

$$\begin{aligned} \frac{\partial \beta}{\partial t} + \chi_R(|U|_{\mathcal{W}}^2) \left[(v \cdot \nabla_H) \beta + \Phi(v) \frac{\partial \beta}{\partial z} + (\eta \cdot \nabla_H) U + \Phi(\eta) \frac{\partial U}{\partial z} + \tilde{G}(\beta) \right] + A\beta \\ + 2\chi'_R(|U|_{\mathcal{W}}^2)(U, \beta)_{\mathcal{W}} (B(U, U) + G(U)) = 0 \end{aligned} \quad (5.51)$$

with initial value $\beta(0) = h$, and

$$\tilde{G}(\beta) = \begin{pmatrix} fk \times \eta - \int_{-1}^z \nabla_H \gamma dz' \\ \Phi(\eta) \end{pmatrix}.$$

We also can rewrite (5.51) in the following form

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \chi_R(|U|_{\mathcal{W}}^2) [(v \cdot \nabla_H) \eta + \Phi(v) \frac{\partial \eta}{\partial z} + (\eta \cdot \nabla_H) v + \Phi(\eta) \frac{\partial v}{\partial z} + fk \times \eta - \int_{-1}^z \nabla_H \gamma dz'] \\ + 2\chi'_R(|U|_{\mathcal{W}}^2)(U, \beta)_{\mathcal{W}} [(v \cdot \nabla_H) v + \Phi(v) \frac{\partial v}{\partial z} + fk \times v + \nabla_H p_b - \int_{-1}^z \nabla_H T dz'] - \Delta \eta = 0, \end{aligned} \quad (5.52)$$

and

$$\begin{aligned} \frac{\partial \gamma}{\partial t} + \chi_R(|U|_{\mathcal{W}}^2) [(v \cdot \nabla_H) \gamma + \Phi(v) \frac{\partial \gamma}{\partial z} + (\eta \cdot \nabla_H) T + \Phi(\eta) \frac{\partial T}{\partial z} + \Phi(\eta)] \\ + 2\chi'_R(|U|_{\mathcal{W}}^2)(U, \beta)_{\mathcal{W}} [(v \cdot \nabla_H) T + \Phi(v) \frac{\partial T}{\partial z} + \Phi(v)] - \Delta \gamma = 0. \end{aligned} \quad (5.53)$$

Multiplying (5.52) by $-\Lambda^{3+4\varepsilon_0} \eta$, then integrating over \mathbb{T}^3 , we have

$$\begin{aligned} & \frac{1}{2} \frac{d|\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^2}{dt} + |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta|^2 \\ = & \chi_R(|U|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} [(v \cdot \nabla_H) \eta + \Phi(v) \frac{\partial \eta}{\partial z} + (\eta \cdot \nabla_H) v + \Phi(\eta) \frac{\partial v}{\partial z}] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \\ & + \chi_R(|U|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} [fk \times \eta - \int_{-1}^z \nabla_H \gamma dz'] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \\ & + 2 \int_{\mathbb{T}^3} \chi'_R(|U|_{\mathcal{W}}^2)(U, \beta)_{\mathcal{W}} [(v \cdot \nabla_H) v + \Phi(v) \frac{\partial v}{\partial z}] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \\ & + 2 \int_{\mathbb{T}^3} \chi'_R(|U|_{\mathcal{W}}^2)(U, \beta)_{\mathcal{W}} [fk \times v + \nabla_H p_b - \int_{-1}^z \nabla_H T dz'] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \\ := & K_1 + K_2 + K_3 + K_4. \end{aligned}$$

For K_1 , we only need to estimate

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} [\Phi(v) \frac{\partial \eta}{\partial z} + \Phi(\eta) \frac{\partial v}{\partial z}] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \right| \\ = & \left| \int_{\mathbb{T}^3} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\Phi(v) \frac{\partial \eta}{\partial z} + \Phi(\eta) \frac{\partial v}{\partial z}] \Lambda^{\frac{5}{2}+2\varepsilon_0} \eta dx dy dz \right| \\ \leq & |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| \left(|\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} \eta| |\Lambda^{1+\sigma_2} v| + |\Lambda^{1+s_1} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0+s_2} v| \right) \\ \leq & |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^2 \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| \\ \leq & |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^{\frac{1}{2}+2\varepsilon_0} |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta|^{\frac{1}{2}-2\varepsilon_0} |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| \\ \leq & \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^{\frac{4}{1+4\varepsilon_0}}, \end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = \frac{1}{2} - 2\varepsilon_0, \quad \sigma_2 = \frac{1}{2} + 2\varepsilon_0, \quad s_1 = 1, \quad s_2 = 0.$$

For K_2 ,

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} [fk \times \eta - \int_{-1}^z \nabla_H \gamma dz'] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \right| \\ & \leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta|^2 + C |\Lambda^{\frac{1}{2}+2\varepsilon_0} \gamma|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^2. \end{aligned}$$

For K_3 ,

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} (U, \beta)_{\mathcal{W}} [\Phi(v) \frac{\partial v}{\partial z}] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \right| \\ & \leq (U, \beta)_{\mathcal{W}} |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| \left(|\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} v| |\Lambda^{\sigma_2} \frac{\partial v}{\partial z}| + |\Lambda^{1+s_1} v| |\Lambda^{\frac{1}{2}+2\varepsilon_0+s_2} \frac{\partial v}{\partial z}| \right) \\ & \leq C(U, \beta)_{\mathcal{W}} |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{\partial v}{\partial z}}| \\ & \leq C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{\partial v}{\partial z}}| + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{\partial v}{\partial z}}| \\ & \leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^4 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^2 (|\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa|^2 + |\Lambda^{\frac{\partial Z_1}{\partial z}}|^2) \\ & \quad + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} T|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma|^2 (|\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa|^2 + |\Lambda^{\frac{\partial Z_1}{\partial z}}|^2), \end{aligned}$$

where $\sigma_1 + \sigma_2 = 1$, $s_1 + s_2 = 1$, and we choose

$$\sigma_1 = 0, \quad \sigma_2 = 1, \quad s_1 = \frac{1}{2} + 2\varepsilon_0, \quad s_2 = \frac{1}{2} - 2\varepsilon_0.$$

For K_4 ,

$$\begin{aligned} & \left| \int_{\mathbb{T}^3} (U, \beta)_{\mathcal{W}} [\int_{-1}^z \nabla_H T dz'] \Lambda^{3+4\varepsilon_0} \eta dx dy dz \right| \\ & = \left| \int_{\mathbb{T}^3} (U, \beta)_{\mathcal{W}} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\int_{-1}^z \nabla_H T dz'] \Lambda^{\frac{5}{2}+2\varepsilon_0} \eta dx dy dz \right| \\ & \leq (U, \beta)_{\mathcal{W}} |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| \\ & \leq C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| \\ & \leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} T|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} T|^4 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma|^2. \end{aligned}$$

Multiplying (5.53) by $-\Lambda^{3+4\varepsilon_0} \gamma$, then integrating over \mathbb{T}^3 , we have

$$\begin{aligned} & \frac{1}{2} \frac{d |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma|^2}{dt} + |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma|^2 \\ & = \chi_R (|U|_{\mathcal{W}}^2) \int_{\mathbb{T}^3} \left[(v \cdot \nabla_H) \gamma + \Phi(v) \frac{\partial \gamma}{\partial z} + (\eta \cdot \nabla_H) T + \Phi(\eta) \frac{\partial T}{\partial z} + \Phi(\eta) \right] \Lambda^{3+4\varepsilon_0} \gamma dx dy dz \\ & \quad + 2 \int_{\mathbb{T}^3} \chi'_R (|U|_{\mathcal{W}}^2) (U, \beta)_{\mathcal{W}} \int_{\mathbb{T}^3} \left[(v \cdot \nabla_H) T + \Phi(v) \frac{\partial T}{\partial z} + \Phi(v) \right] \Lambda^{3+4\varepsilon_0} \gamma dx dy dz \\ & := K_5 + K_6. \end{aligned}$$

For K_5 , we only need to estimate the following term,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} [\Phi(v) \frac{\partial \gamma}{\partial z} + \Phi(\eta) \frac{\partial T}{\partial z}] \Lambda^{3+4\varepsilon_0} \gamma dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\Phi(v) \frac{\partial \gamma}{\partial z} + \Phi(\eta) \frac{\partial T}{\partial z}] \Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma dx dy dz \right| \\
&\leq C |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| (|\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^2 \gamma| + |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| |\Lambda^2 \eta|) \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \eta|^2 + \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^{\frac{4}{1+4\varepsilon_0}} + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} T|^{\frac{4}{1+4\varepsilon_0}}.
\end{aligned}$$

For K_6 ,

$$\begin{aligned}
& \left| \int_{\mathbb{T}^3} \chi'_R(|U|_{\mathcal{W}}^2)(U, \beta)_{\mathcal{W}} \int_{\mathbb{T}^3} [\Phi(v) \frac{\partial T}{\partial z}] \Lambda^{3+4\varepsilon_0} \gamma dx dy dz \right| \\
&= \left| \int_{\mathbb{T}^3} \chi'_R(|U|_{\mathcal{W}}^2)(U, \beta)_{\mathcal{W}} \int_{\mathbb{T}^3} \Lambda^{\frac{1}{2}+2\varepsilon_0} [\Phi(v) \frac{\partial T}{\partial z}] \Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma dx dy dz \right| \\
&\leq C(U, \beta)_{\mathcal{W}} |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| (|\Lambda^{\frac{3}{2}+2\varepsilon_0+\sigma_1} v| |\Lambda^{1+\sigma_2} T| + |\Lambda^{1+s_1} v| |\Lambda^{\frac{1}{2}+2\varepsilon_0+s_2} \frac{\partial T}{\partial z}|) \\
&\leq C(U, \beta)_{\mathcal{W}} |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{\partial T}{\partial z}}| \\
&\leq C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{\partial T}{\partial z}}| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma| |\Lambda^{\frac{\partial T}{\partial z}}| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| \\
&\leq C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{5}{2}+2\varepsilon_0} g| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma| |\Lambda^{\frac{5}{2}+2\varepsilon_0} g| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| \\
&\quad + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta| |\Lambda^{\frac{\partial Z_2}{\partial z}}| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v| |\Lambda^{\frac{3}{2}+2\varepsilon_0} T| |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma| |\Lambda^{\frac{\partial Z_2}{\partial z}}| |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma| \\
&\leq \varepsilon |\Lambda^{\frac{5}{2}+2\varepsilon_0} \gamma|^2 + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^4 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \eta|^2 (|\Lambda^{\frac{5}{2}+2\varepsilon_0} g|^2 + |\Lambda^{\frac{\partial Z_2}{\partial z}}|^2) \\
&\quad + C |\Lambda^{\frac{3}{2}+2\varepsilon_0} v|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} T|^2 |\Lambda^{\frac{3}{2}+2\varepsilon_0} \gamma|^2 (|\Lambda^{\frac{5}{2}+2\varepsilon_0} g|^2 + |\Lambda^{\frac{\partial Z_2}{\partial z}}|^2).
\end{aligned}$$

Combing all the above estimations, we have for $|U|_{\mathcal{W}}^2 \leq R$,

$$\frac{d|\beta|_{\mathcal{W}}^2}{dt} + \|\beta\|_{\frac{5}{2}+2\varepsilon_0}^2 \leq C \left(C(R) + |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa|^2 + |\Lambda^{\frac{5}{2}+2\varepsilon_0} g|^2 + |\Lambda^{\frac{\partial Z_1}{\partial z}}|^2 + |\Lambda^{\frac{\partial Z_2}{\partial z}}|^2 \right) |\beta|_{\mathcal{W}}^2,$$

by Gronwall's inequality and (5.43), we finally get

$$\begin{aligned}
& \int_0^t \|\beta(l)\|_{\frac{5}{2}+2\varepsilon_0}^2 dl \\
&\leq C |h|_{\mathcal{W}}^2 + \exp \left(C \int_0^t \left(C(R) + |\Lambda^{\frac{5}{2}+2\varepsilon_0} \kappa|^2 + |\Lambda^{\frac{5}{2}+2\varepsilon_0} g|^2 + |\Lambda^{\frac{\partial Z_1}{\partial z}}|^2 + |\Lambda^{\frac{\partial Z_2}{\partial z}}|^2 \right) dl \right) |h|_{\mathcal{W}}^2 \\
&\leq C |h|_{\mathcal{W}}^2 + \exp \left(C \left(|\Lambda^{\frac{3}{2}+2\varepsilon_0} y|^2 + \int_0^t \left(C(R) + |\Lambda^{\frac{\partial Z_1}{\partial z}}|^2 + |\Lambda^{\frac{\partial Z_2}{\partial z}}|^2 \right) dl \right) \right) |h|_{\mathcal{W}}^2.
\end{aligned}$$

Since Z_i is a Gaussian random variable in $C([0, \infty), D(\Lambda^{2+2\varepsilon_0-2\varepsilon}))$, for every $\varepsilon > 0$, by Fernique's theorem, we could choose t_0 small enough and obtain

$$\mathbb{E} \int_0^{t_0} \|\beta(l)\|_{\frac{5}{2}+2\varepsilon_0}^2 dl \leq C(t_0, R) |h|_{\mathcal{W}}^2.$$

From **Hypothesis H1**, it follows that

$$Q^{-\frac{1}{2}} = Q_0^{-\frac{1}{2}} A^{\frac{5}{4}+\varepsilon_0},$$

thus, the assertion of (5.50) holds for t_0 . For general t , by the semigroup property, the assertion follows easily. ■

Remark 6. *In order to apply Fernique's theorem to Gaussian process Z , we have to make use of $\Lambda \frac{\partial Z}{\partial z}$ and control its power to be less than 2 during the estimate of K_3 and K_6 .*

Proof of Theorem 1.2 Theorem 5.1 and Proposition 5.2 imply the result by Theorem 5.4 in [8].

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